



Morrison, Ewan K. (2012) Modular Frobenius Manifolds. PhD thesis

<http://theses.gla.ac.uk/4260/>

Copyright and moral rights for this thesis are retained by the author

A copy can be downloaded for personal non-commercial research or study, without prior permission or charge

This thesis cannot be reproduced or quoted extensively from without first obtaining permission in writing from the Author

The content must not be changed in any way or sold commercially in any format or medium without the formal permission of the Author

When referring to this work, full bibliographic details including the author, title, awarding institution and date of the thesis must be given.

Modular Frobenius Manifolds

A thesis submitted for the degree of
Doctor of Philosophy
at the
College of Science & Engineering
University of Glasgow

© Ewan K. Morrison
2012

Introduction

Many physical processes may be modelled mathematically using a differential equation. For example, Newton's second law tells us that the position of a point-like particle in a gravitational field is governed by a second order ordinary differential equation. Many years later Hamilton gave a more geometric understanding of Newton's equations of classical mechanics by looking at solution curves in the position-momentum space. This in turn led to the discovery of symplectic geometry. The beautiful theorem of Liouville then gives the dictionary to translate the solvability of Hamilton's equations into mathematics: the existence of a maximal set of Poisson commuting invariants.

Differential equations come in many flavours - linear and nonlinear, ordinary and partial, evolutionary and otherwise. The above example is an archetype of an integrable differential equation: one's initial motivation may come from physics, but with understanding of its solutions comes a flurry of pure mathematics.

Central to this thesis are the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations, which are a system of over-determined partial differential equations that were found in the study of topological field theory in the early '90's [13, 67]. They are named after Edward Witten, Robbert Dijkgraaf, Erik Verlinde, and Herman Verlinde who are pioneers in this area. Roughly, they describe the conditions for the third derivatives

$$c_{\alpha\beta\gamma}(\mathbf{t}) := \frac{\partial^3 F(\mathbf{t})}{\partial t_\alpha \partial t_\beta \partial t_\gamma}$$

of some function $F(\mathbf{t})$ to define the structure functions of an algebra that is commutative, associative, and unital.

The WDVV equations are an integrable system: they admit a zero-curvature representation via the deformed Euclidean, or *Dubrovin* connection. Perhaps one of the most spectacular classes of solutions from a mathematical perspective are generating functions of genus zero Gromov-Witten invariants of a symplectic manifold. That these generating functions should satisfy WDVV follows from the geometry of the moduli spaces of stable maps from the Riemann sphere with some marked points into our symplectic manifold. The structure functions of the algebra then define a deformation of the cohomology ring of the symplectic manifold, known as the *quantum cohomology* ring. For example,

consider complex projective space of dimension two, whose small quantum cohomology ring is isomorphic to $\mathbb{C}[x]/\langle x^3 - e^{-t_2} = 0 \rangle$. The corresponding solution to WDVV (a formal power series) is

$$F(t_1, t_2, t_3) = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + \sum_{d \geq 1} \frac{N_d t_3^{3d-1}}{(3d-1)!} e^{t_2 d},$$

where the coefficients satisfy the recursion relation

$$N_d = (3d-4)! \sum_{a+b=d} \frac{a^2 b(3b-1)(2a-b)}{(3a-1)!(3b-1)!} N_a N_b.$$

Thus by just knowing $N_1 = 1$, $N_2 = 1$, one can determine N_d , for all $d \geq 0$. The first few are

$$1, \quad 1, \quad 12, \quad 620, \quad 87304, \quad 26312976, \quad \dots$$

The numbers N_d have the interpretation of being the number of rational curves of degree d passing through $3d-1$ points in \mathbb{P}^2 .

As mentioned above, the WDVV equations first appeared in topological field theory, which is one model that can facilitate the marriage between quantum field theory and gravity. Another approach is via a so-called matrix model. In the latter approach, it turns out that the partition function of the theory is a tau-function of the Korteweg-de Vries (KdV) equation,

$$u_t = 6uu_x + u_{xxx},$$

which is another example of an integrable non-linear differential equation. Given that any correct theory of quantum gravity must be unique, Witten was lead to conjecture [67] that the generating function of the Gromov-Witten invariants (in this case of a point) must also be a tau function for the KdV hierarchy. This was later proved by Kontsevich [38], leading to him being awarded the Fields Medal in 1998. In fact, this tau-function has a power series expansion, with the zeroth order term (the generating function of the genus zero invariants) satisfying the WDVV equations. This zeroth order term is also a tau function of the *dispersionless* KdV equation.

The notion of a Frobenius manifold was introduced by Dubrovin [17] in order to provide a geometric understanding of the solutions to the WDVV equations. Further, the *principal hierarchy* of a Frobenius manifold shows how to construct an integrable hierarchy of partial differential equations of hydrodynamic type (like the dispersionless limit of the KdV equation) from a solution to the WDVV equations. This is done by constructing a pair of compatible Poisson brackets, and an infinite number of conserved quantities on the loop space of the Frobenius manifold. This hierarchy is integrable in the sense that this infinite family of conserved quantities is in involution with respect to both Poisson brackets.

This thesis studies how a symmetry defined on the solution space to the WDVV equations, called the *inversion symmetry*, singles out a special class of solution: those that lie at its fixed points. We will learn how demanding invariance of the solution under this symmetry forces it to take on a very rich form: that of a quasi-modular function. We will also study how the corresponding principal hierarchies inherit this symmetry from the solution to the WDVV equations.

More specifically, Chapter 1 is introductory material on the theory of Frobenius manifolds. We provide motivation, basic examples and tools that will be useful to us later. As such it contains no original material. The references [1, 16, 17, 18, 19, 35, 37] were extremely useful in the preparation of this chapter.

Chapter 2 is based on [45], which was written in collaboration with Professor Ian Strachan. It appeared in *Physica D: Nonlinear Phenomena*. We study those solutions to WDVV that lie at the fixed points the inversion symmetry. By studying the transformation and homogeneity properties that such solutions must have, we set out a program for classification, with complete results presented for dimensions three and four, together with partial results for dimension five. We show how various examples that have appeared in the literature fit into our framework. Any material here that is not original is clearly referenced.

Chapter 3 is background material on integrable systems. It begins with a little history of the KdV equation, and explores some of the key properties that make it interesting [15, 3]. We move on to the construction of Poisson brackets on loop spaces [48, 24], and more specifically Poisson brackets of hydrodynamic type. We then sketch Dubrovin's construction [16] of the principal hierarchy from the geometry of the Frobenius manifold.

Chapter 4 is based on [44], a joint work with Professor Ian Strachan which appeared in *International Mathematics Research Notices*. We study how the inversion symmetry defined on the solution space to the WDVV equations lifts to the principal hierarchy. It turns out that the action is an example of a so-called *reciprocal* transformation, introduced by Rogers [52]. These results were obtained independently in [41], and more recently in [26]. There is also some background material which has been tailored from Dubrovin's work [20] on so-called *almost dual* solutions to WDVV. We give a definition of the inversion symmetry for these solutions and study how the inversion symmetry acts on the associated hydrodynamic systems.

Chapter 5 is a very natural continuation of the results obtained in Chapter 4. Motivated by the Witten-Kontsevich theorem, Dubrovin & Zhang [23] showed how the inclusion of the elliptic Gromov-Witten invariants into the tau-function perturbs the equations of the hierarchy. The results of Chapter 5 show how to extend the symmetry found at the level of the hydrodynamic equations to these first order, (or *genus one*), deformations.

Acknowledgments

First and foremost I would like to thank my supervisor, Ian Strachan, for his guidance, encouragement and enthusiasm over the past three and a half years. I would like to thank Tarig Abdel Gadir for numerous interesting mathematical encounters, and making my time in Glasgow so enjoyable. Thanks to my office mates, Steven O'Hagan, Joe Mullaney, Fawad Hussain, Beibei Lei, and Yujue Hao, for making both 524 and 321A great places to work. Thanks to team Times2 puzzles page. Thanks to Alexander Quintero-Velez for taking the time to answer so many of my questions, and many interesting conversations. I would like to thank Paul Norbury, Arun Ram and the Department of Mathematics and Statistics at the University of Melbourne, where part of the work included in this thesis was completed.

I would like to thank the School of Mathematics and Statistics at the University of Glasgow, in particular the Integrable Systems and Mathematical Physics research group, for providing a supportive and stimulating research environment. Thanks also to the Engineering and Physical Sciences Research Council, the London Mathematical Society, Glasgow Mathematical Journal Trust, the Royal Society, the Institute of Physics, the International Centre for Mathematical Sciences, and the UK integrable systems and mathematical physics community as a whole for both the funding and arrangement of so many interesting summer schools and conferences.

The support of my parents, brother, and sister has been indispensable in completion of this thesis.

Contents

1	Frobenius Manifolds	1
1.1	Frobenius Algebras	2
1.2	Frobenius Manifolds	7
1.3	Flat Sections of The Dubrovin Connection	12
1.4	Semi-Simple Frobenius Manifolds	19
1.4.1	The Intersection Form	25
1.4.2	Flat Pencils of Metrics and Frobenius Manifolds	29
1.5	The Inversion Symmetry	32
1.6	Examples of Frobenius Manifolds	37
1.6.1	Rational Maps of Degree $N + 1$	37
1.6.2	Coxeter Group Orbit Spaces	40
2	Modular Frobenius Manifolds	45
2.1	Modular Frobenius Manifolds	46
2.2	Modularity and Quasi-Homogeneity	48
2.3	The WDVV Equations and Modular Dynamical Systems	53
2.3.1	Modular Dynamical Systems for $N = 3$	55
2.3.2	Modular Dynamical Systems for $N = 4$	56
2.3.3	Modular Dynamical Systems for $N = 5$	64
2.4	Solving Modular Dynamical Systems	66
2.5	Modular Dynamical Systems and Non-Associative Algebras	74
3	From Frobenius Manifolds to Integrable Systems	78
3.1	Infinite Dimensional Hamiltonian Systems	79
3.1.1	The Korteweg-de Vries Equation	79
3.1.2	Conservation Laws of the KdV Equation	80
3.1.3	Poisson Brackets on Loop Spaces	81
3.1.4	Poisson Brackets of Hydrodynamic Type	83
3.1.5	Bi-Hamiltonian Structures and Flat Pencils of Metrics	85
3.2	The Principal Hierarchy	86

3.2.1 Families of Commuting Functionals on Frobenius Manifolds	86
3.2.2 The Principal Hierarchy	87
4 The Inversion Symmetry and Principal Hierarchies	91
4.1 The Inversion Symmetry in Canonical Coordinates	92
4.2 Inversion Symmetry and Principal Hierarchies	94
4.2.1 Dispersionless Limits of the David Benney and Harry Dym Hierarchies	101
4.2.2 A Modular Integrable System	103
4.3 Almost Duality of Frobenius Manifolds	106
4.3.1 The Existence of F^* and Twisted Periods	106
4.3.2 Almost duality for $d \neq 1$	108
4.3.3 Almost Duality for $d = 1$	112
4.4 Inversion Symmetry and Almost Duality	113
4.5 The Inversion Symmetry and Twisted Periods	120
5 The Inversion Symmetry at Genus One	123
5.1 Introduction	124
5.2 Preliminaries	127
5.2.1 Four-point Functions in Canonical Coordinates	127
5.3 Action on Deformed Hamiltonians	128
5.3.1 Perturbation 1: $\delta \mathcal{H}'_{(n,\kappa)}$	128
5.3.2 Perturbation 2: $\delta \mathcal{H}''_{(n,\kappa)}$	131
5.4 Action on Deformed Poisson Prackets	133
5.4.1 Deformations of the Poisson Brackets in Canonical Coordinates	133
5.4.2 The Inversion Symmetry and the Deformed Poisson Brackets	136
5.4.3 The Action on the Sections $\nabla_X dh_{\alpha,p}$ and $\nabla_X^2 dh_{\alpha,p}$	143
5.5 The Inversion Symmetry and the Principal Hierarchy at Genus One	148

Chapter 1

Frobenius Manifolds

1.1 Frobenius Algebras

Frobenius algebras are particularly rich mathematical structures that arise in different areas of mathematics, as well as computer science. This is primarily because, as we will see, they encode topological information, though they are classically elegant algebraic structures in their own right. They are named after the German mathematician Ferdinand Georg Frobenius. There are several definitions, but the one best suited to our needs is:

Definition 1. *The triple (A, \circ, η) is a Frobenius algebra if:*

1. (A, \circ) is a commutative associative algebra over \mathbb{C} with unity e ;
2. $\eta : A \otimes A \rightarrow \mathbb{C}$ is a non-degenerate bilinear pairing;
3. The pairing η and multiplication \circ satisfy the following Frobenius condition

$$\eta(X \circ Y, Z) = \eta(X, Y \circ Z), \quad X, Y, Z \in A.$$

The bilinear pairing η is often called the ‘Frobenius form’.

Frobenius algebras arise in a purely abstract algebraic manner:

Example 1. *Let A be the space of $n \times n$ matrices over \mathbb{C} , with respect to the usual multiplication of matrices. Define $\eta(x, y) = \text{tr}(xy)$, for $x, y \in A$. Because tr is a linear map, it follows that η is bilinear. To prove that η is non-degenerate, let e_{ij} be the matrix with 1 in the $(i, j)^{\text{th}}$ -entry, and zero everywhere else. Then $\eta(x, y) = 0, \forall x$ implies, in particular, that*

$$0 = \text{tr}(e_{ij}y) = y_{ji}, \quad \text{for } 1 \leq i, j \leq n.$$

That is $y = 0$. The compatibility between the multiplication and the Frobenius form follows from associativity of matrix multiplication.

They are important in singularity theory:

Example 2. *Let $p : \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial with an isolated singularity at $z = 0$. This means that*

$$\begin{cases} dp(z) = 0 & \text{for } z = 0, \\ dp(z) \neq 0 & \text{for } z \neq 0 \text{ sufficiently small.} \end{cases}$$

Let

$$A = \frac{\mathbb{C}[z]}{\langle dp = 0 \rangle}.$$

The algebra A is known as the Jacobi ring, or local algebra of the singularity [4]. We define, for $f, g \in A$,

$$\eta(f, g) = -\text{Res}_{z=0} \left(\frac{fg}{\frac{\partial p}{\partial z_1} \frac{\partial p}{\partial z_2} \cdots \frac{\partial p}{\partial z_n}} \right) dz.$$

For example, for $n = 1$, taking

$$p = z^{N+1},$$

we have

$$\eta(f, g) = \frac{1}{N+1} \left(\text{Coefficient of } z^{N-1} \text{ in the product } fg \right).$$

They are integral to cohomological field theories, such as string theories:

Example 3. Let \mathcal{M} be a smooth, orientable manifold of dimension $2n$. Take as A the evenly graded cohomology of \mathcal{M} (the wedge product is commutative over even cohomology classes). For $\alpha, \beta \in H^{ev}(\mathcal{M})$, define

$$\eta(\alpha, \beta) = \int_{\mathcal{M}} \alpha \wedge \beta,$$

that is the intersection form on α and β . The Frobenius property follows from associativity of the wedge product,

$$\eta(\alpha \circ \beta, \gamma) = \int_{\mathcal{M}} (\alpha \wedge \beta) \wedge \gamma = \int_{\mathcal{M}} \alpha \wedge (\beta \wedge \gamma) = \eta(\alpha, \beta \circ \gamma),$$

while Poincarè duality ensures η is non-degenerate. Particularly, let $\mathcal{M} = \mathbb{P}^d$, complex projective space of complex dimension d . The even cohomology $H^{ev}(\mathbb{P}^d)$ is spanned by $\{1, \omega, \omega^2, \dots, \omega^d\}$, where ω is the standard Kähler form on \mathbb{P}^d normalised by

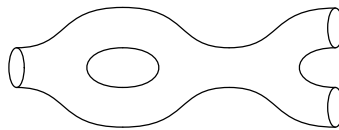
$$\int_{\mathbb{P}^d} \omega^d = 1.$$

On this basis, the bilinear form reads

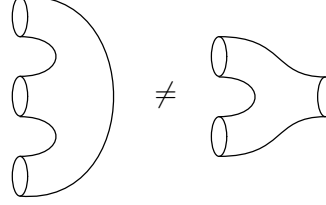
$$\eta(\omega^i, \omega^j) = \delta_{i+j, d}.$$

The next example explores one of the reasons that interest in Frobenius algebras has intensified in recent years, and is closer to Dubrovin's original motivation. In his 1988 work 'Topological quantum field theories', Atiyah [1] presented an axiomatic approach to topological field theories. It was later proved by Dijkgraaf [12] that when these topological field theories are two dimensional (these are more commonly called topological string theories), they are equivalent to Frobenius algebras.

Some notational points: Let $2Cob$ denote the category whose objects are oriented, closed 1-manifolds, and whose morphisms are given by oriented cobordisms. For example, a morphism between S^1 and the disjoint union of two copies of S^1 might look like



The orientation on the cobordism is important: if the orientation of a boundary S^1 is anti-clockwise, we put it on the left of the picture, or as an ‘input’. If the orientation is clockwise, we put it on the right as an ‘output’. So, for example,



Isomorphism classes will therefore consist of homeomorphic surfaces with the same number of input and output copies of S^1 .

Let $\text{Vect}_{\mathbb{C}}$ denote the category of complex, finite dimensional vector spaces. So the objects are vector spaces, and the morphisms are \mathbb{C} -linear maps.

We will give the definition of a two-dimensional topological field theory due to Atiyah [1], although the way in which we explore the axioms here is closer to the approach given by Hitchin in a series of lectures given on Frobenius manifolds [35].

Definition 2. A two-dimensional topological field theory (2dTQFT) is a functor $Z : 2\text{Cob} \rightarrow \text{Vect}_{\mathbb{C}}$, which assigns to each oriented 1-manifold Σ a complex finite dimensional vector space $Z(\Sigma)$, and to each oriented 2-manifold \mathcal{M} with $\Sigma = \partial\mathcal{M}$ a vector $Z(\mathcal{M}) \in Z(\Sigma)$, subject to the following conditions:

1. $Z(\Sigma^-) = Z(\Sigma)^*$, where Σ^- denotes the manifold Σ with the opposite orientation,
2. $Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$,
3. if $\partial\mathcal{M}_1 = \Sigma_1 \cup \Sigma_2$, and $\partial\mathcal{M}_2 = \Sigma_2 \cup \Sigma_3$, then $Z(\mathcal{M}_1 \cup_{\Sigma_2} \mathcal{M}_2) \in \text{Hom}(\Sigma_1, \Sigma_3)$,
4. if $\partial\mathcal{M}_1 = \partial\mathcal{M}_2 \cup X \cup X^-$, and \mathcal{M}_1 and \mathcal{M}_2 coincide outside a ball, then 1. and 2. imply $Z(\mathcal{M}_1) = Z(\mathcal{M}_2) \otimes Z(X) \otimes Z(X)^*$. We identify the gluing of \mathcal{M}_1 and \mathcal{M}_2 along X with the natural pairing $Z(X) \otimes Z(X)^* \rightarrow \mathbb{C}$ in the tensor product.
5. $Z(\emptyset) = \mathbb{C}$,
6. $Z(\Sigma \times I) = \text{id} \in \text{Hom}(Z(\Sigma), Z(\Sigma))$.

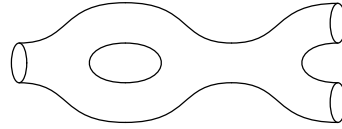
The definition of $Z(\mathcal{M})$ may appear at first sight vexatious: in order for Z to be a functor, $Z(\mathcal{M})$ must be a linear map. Note that because of the dual pairing of vector spaces it is enough to define $Z(\mathcal{M})$ where \mathcal{M} has an empty boundary on the left or right, $\partial\mathcal{M} = \emptyset \cup \Sigma$. In this case, it follows from axioms 2. and 5. that $Z(\mathcal{M}) : \mathbb{C} \rightarrow Z(\Sigma)$ and is hence associated with a single vector.

Axioms 1. and 2. say that if $\partial\mathcal{M} = \Sigma_1 \cup \Sigma_2$, then $Z(\mathcal{M}) \in \text{Hom}(Z(\Sigma_1), Z(\Sigma_2))$. Therefore a cobordism defines a linear map $Z(\mathcal{M}) : Z(\Sigma_1) \rightarrow Z(\Sigma_2)$. Composing two such cobordisms gives axiom 3. Functoriality, together with axiom 6. imply homotopy invariance.

Let $A = Z(S^1)$, and $\{\phi_1 \dots \phi_N\}$ be a basis of A . We will endow A with the structure of a Frobenius algebra as follows. Because S^1 is the unique closed connected 1-manifold, the image of any surface is an element of $\underbrace{A \otimes \dots \otimes A}_k \otimes \underbrace{A^* \otimes \dots \otimes A^*}_s$. The components of the images of the surfaces define the genus- g correlation functions of the field theory,

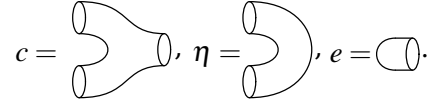
$$\langle \phi_{\alpha_1}, \dots, \phi_{\alpha_k}, \phi^{\alpha_{k+1}}, \dots, \phi^{\alpha_{k+j}} \rangle_g := v_{\alpha_{k+1} \dots \alpha_{k+j}, g}^{\alpha_1 \dots \alpha_k}.$$

For example,



$$\xrightarrow{Z} \sum_{i,j,k=1}^N v_{jk,1}^i \phi_i \otimes \phi^j \otimes \phi^k \in A \otimes A^* \otimes A^*.$$

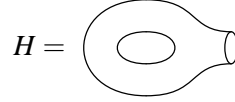
Let us define some genus zero correlation functions by the surfaces



$$c = \text{pair of pants}, \quad \eta = \text{cup}, \quad e = \text{cylinder}.$$

So, $c \in A^* \otimes A^* \otimes A$, $\eta \in A^* \otimes A^*$, and $e \in A$ (note that strictly speaking we mean the images of these surfaces under Z , but we write it like this for notational convenience). We are now ready to state the main

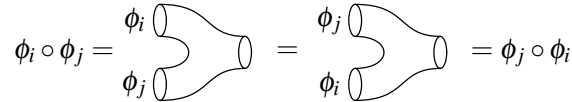
Theorem 1. [12] *The tensors c and η define on A the structure of a Frobenius algebra with unity e . Further, if*



$$H = \text{torus with boundary}.$$

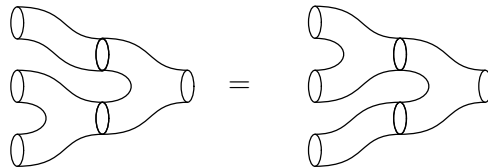
then $\langle \phi_{\alpha_1} \dots \phi_{\alpha_N} \rangle_g = \eta(\phi_{\alpha_1} \circ \dots \circ \phi_{\alpha_N}, \underbrace{H \circ \dots \circ H}_g)$, where on the right hand side \circ denotes multiplication in the Frobenius algebra.

Proof. The first step will be to confirm that the images of the surfaces defined above satisfy the axioms of a Frobenius algebra. Commutativity of multiplication follows from



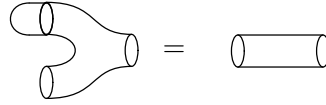
$$\phi_i \circ \phi_j = \text{pair of pants with } \phi_i \text{ on top} = \text{pair of pants with } \phi_j \text{ on top} = \phi_j \circ \phi_i$$

since there clearly exists a homeomorphism that interchanges the two input S^1 's while fixing the output. Associativity follows from

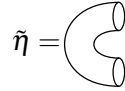


$$\text{pair of pants with } (\phi_i, \phi_j) \text{ on top} = \text{pair of pants with } \phi_i \text{ on top and } (\phi_j, \phi_k) \text{ on top}$$

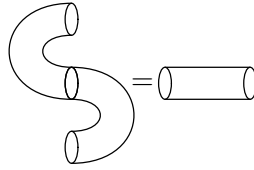
The surface e indeed provides the unity:



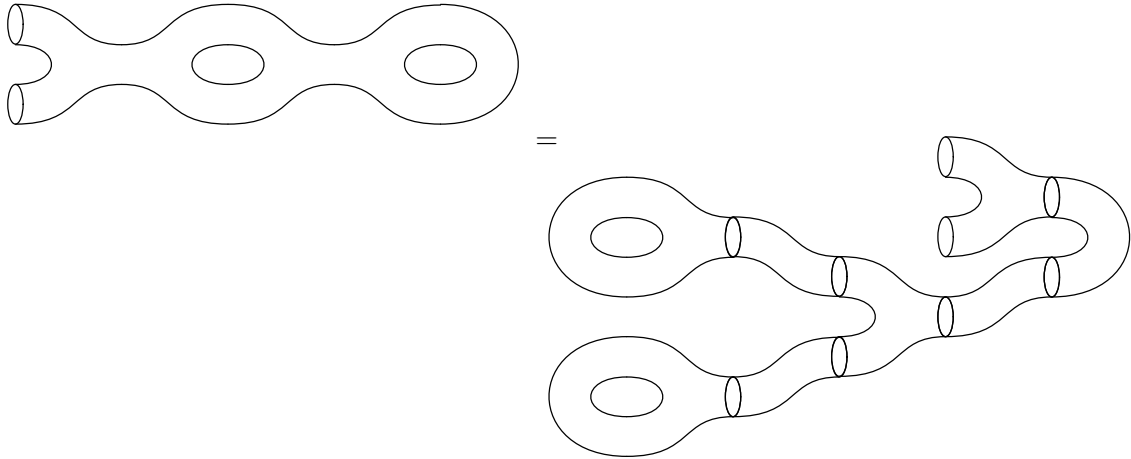
To see that the metric is non-degenerate, consider the surface $\tilde{\eta} : \mathbb{C} \rightarrow A^* \otimes A^*$:



Then $Z(\tilde{\eta} \cup_{S^1} \eta) = \text{id} : A \rightarrow A$:



So $\tilde{\eta} = \eta^{-1}$. The second part of the theorem follows from the diagram



which demonstrates the relation $\langle \phi_{\alpha_1} \phi_{\alpha_2} \rangle_2 = \eta(\phi_{\alpha_1} \circ \phi_{\alpha_2}, H^2)$ □

In fact it can be proved [37] that the surfaces e, c , and η are sufficient to generate any morphism of $2Cob$. This means that the category of commutative Frobenius algebras is *equivalent* to the category of $2dTQFTs$.

Now that we have defined a Frobenius algebra, and provided some motivation as to why they are interesting objects, let us turn our attention to the definition of a Frobenius manifold. Roughly speaking, a Frobenius manifold is a manifold for which each tangent space carries the structure of a Frobenius algebra. The structure of these algebras should vary smoothly as one moves between points in the underlying manifold. From the point of view of $2dTQFTs$, they parameterise certain canonical families of theories.

1.2 Frobenius Manifolds

Frobenius manifolds were introduced by Boris Dubrovin (see, for example [16, 17]) as a geometric interpretation of the solutions to the following nonlinear systems of partial differential equations

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\delta \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\alpha \partial t^\gamma} \quad (1.2.1)$$

for some function quasihomogeneous function $F(\mathbf{t})$. These equations are known as the WDVV equations after Edward Witten, Robbert Dijkgraaf, Erik Verlinde, and Herman Verlinde who are pioneers in the area of topological field theory, where the system (1.2.1) was first discovered. The indices here take values in $\{1, \dots, N\}$. When using Greek indices, use of the Einstein summation convention will be assumed unless otherwise stated. The array $\eta^{\alpha\beta}$ is defined by $\eta^{\alpha\beta} \eta_{\beta\kappa} = \delta_\kappa^\alpha$ where

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} \quad (1.2.2)$$

is constant and non-degenerate. Demanding F be quasihomogenous means that there exists scalars d_1, \dots, d_N, d_F such that

$$F(\lambda^{d_1} t^1, \dots, \lambda^{d_N} t^N) = \lambda^{d_F} F(t^1, \dots, t^N), \quad \forall \lambda \in \mathbb{R}. \quad (1.2.3)$$

Note that if all the d_i are equal, this property restricts to homogeneity. Observe also that solutions of the system (1.2.1) are only defined up to the addition of a quadratic. The homogeneity property, together with this quadratic ambiguity may be re-cast as

$$\mathcal{L}_E F = E(F) = d_F F \quad \text{modulo quadratic terms.}$$

by introducing an Euler field,

$$E = \sum_{\alpha} (d_{\alpha} t^{\alpha} + r^{\alpha}) \partial_{\alpha}.$$

The rich geometric object underlying this ansatz for the WDVV equations is known as a *Frobenius manifold*.

Definition 3. Let \mathcal{M} be a smooth manifold. \mathcal{M} is called a Frobenius manifold if each tangent space $T_t \mathcal{M}$ is equipped with the structure of a Frobenius algebra varying smoothly with $t \in \mathcal{M}$, and further

1. The invariant inner product η defines a flat metric on \mathcal{M} .
2. The unity vector field is covariantly constant with respect to the Levi-Civita connec-

tion for η ,

$$\nabla e = 0. \quad (1.2.4)$$

3. Let

$$c(X, Y, Z) := \eta(X \circ Y, Z), \quad X, Y, Z \in T_t \mathcal{M}. \quad (1.2.5)$$

Then the $(0, 4)$ tensor $\nabla_{\mathcal{W}} c(X, Y, Z)$ is totally symmetric.

4. There exists a vector field $E \in \Gamma(T\mathcal{M}, \mathcal{M})$ such that $\nabla \nabla E = 0$ and

$$\mathcal{L}_E \eta = (2 - d)\eta, \quad \mathcal{L}_E \circ = \circ, \quad \mathcal{L}_E e = -e. \quad (1.2.6)$$

E is called the Euler vector field.

Let us recover the ansatz outlined above for the WDVV equations from the definition of a Frobenius manifold. Condition 1 implies there exists a choice of coordinates (t^1, \dots, t^N) such that the Gram matrix $\eta_{\alpha\beta} = (\partial_\alpha, \partial_\beta)$ is constant. Furthermore, this may be done in such a way that $e = \partial_1$. In such a coordinate system, partial and covariant derivatives coincide, and condition 3 becomes

$$c_{\alpha\beta\gamma, \kappa} = c_{\alpha\beta\kappa, \gamma}.$$

Successive applications of the Poincaré lemma then implies local existence of a function $F(t)$ called the *free energy* of the Frobenius manifold such that

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}. \quad (1.2.7)$$

Since $\eta(X, Y) = \eta(e \circ X, Y) = c(e, X, Y)$, we have

$$\eta_{\alpha\beta} = c_{1\alpha\beta}. \quad (1.2.8)$$

Defining $(\eta_{\alpha\beta})^{-1} = \eta^{\alpha\beta}$, the components of \circ are given by $c_{\beta\gamma}^\alpha = \eta^{\alpha\varepsilon} c_{\varepsilon\beta\gamma}$. Associativity of \circ is then equivalent to the system (1.2.1). Turning our attention to axiom 4.,

$$\nabla \nabla E = 0 \Rightarrow E = (q_\beta^\alpha t^\beta + r^\alpha) \partial_\alpha,$$

for some constants q_β^α, r^α . Note that

$$\left. \begin{array}{l} \mathcal{L}_E \eta = (2 - d)\eta \\ \mathcal{L}_E \circ = \circ \end{array} \right\} \Rightarrow \mathcal{L}_E c = (3 - d)c.$$

In the flat coordinate system,

$$\mathcal{L}_E(\partial_\alpha \partial_\beta \partial_\gamma F) = E(\partial_\alpha \partial_\beta \partial_\gamma F) - q_\gamma^\kappa \partial_\kappa \partial_\alpha \partial_\beta F - q_\beta^\kappa \partial_\kappa \partial_\alpha \partial_\gamma F - q_\alpha^\kappa \partial_\kappa \partial_\gamma \partial_\beta F = \partial_\alpha \partial_\beta \partial_\gamma (E(F)).$$

So

$$\mathcal{L}_E c = (3-d)c \Rightarrow \partial_\alpha \partial_\beta \partial_\gamma (E(F) - (3-d)F) = 0 \Rightarrow E(F) = (3-d)F + A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C. \quad (1.2.9)$$

If $Q = (q_\beta^\alpha)$ is diagonalisable, there exists another system of flat coordinates (recall that flat coordinates are only defined up to a linear transformation) such that

$$E = \sum_{\alpha=1}^N (d_\alpha t^\alpha + r^\alpha) \partial_\alpha$$

where $\{d_1 \dots d_N\} = \text{spec}(Q)$. The relation $\mathcal{L}_E e = -e \Rightarrow q_1^\sigma = \delta_1^\sigma$, so any such linear change of the coordinates preserves $e = \partial_1$. By shifting $t^\alpha \rightarrow t^\alpha - r^\alpha$, for those α such that $d_\alpha \neq 0$, we obtain

$$E = \sum_{\alpha=1}^N d_\alpha t^\alpha \partial_\alpha + \sum_{\alpha: d_\alpha=0} r^\alpha \partial_\alpha, \quad (1.2.10)$$

which is the homogeneity condition (1.2.3) in terms of the Euler field. We will restrict to the case where the matrix Q is diagonalisable over $T\mathcal{M} \setminus \text{Ker} Q$.

Let us present a technical lemma that will allow us to obtain some more explicit formulas.

Note that because $\mathcal{L}_E e = -e$, we have

$$\begin{aligned} \mathcal{L}_E \eta \left(\frac{\partial}{\partial t^1}, \frac{\partial}{\partial t^1} \right) &= (\mathcal{L}_E \eta) \left(\frac{\partial}{\partial t^1}, \frac{\partial}{\partial t^1} \right) + 2\eta \left(\mathcal{L}_E \frac{\partial}{\partial t^1}, \frac{\partial}{\partial t^1} \right) \\ &= -d\eta \left(\frac{\partial}{\partial t^1}, \frac{\partial}{\partial t^1} \right) \end{aligned}$$

because $\mathcal{L}_E \eta = (2-d)\eta^\dagger$. By definition the function η_{11} is constant in the flat coordinates, and so either $\eta_{11} = 0$, or $d = 0$.

Lemma 1. [16] *If $\eta_{11} = 0$ and Q is diagonalisable, then by a (possibly complex) linear change of coordinates t^α , we can reduce the Gram matrix $\eta_{\alpha\beta}$ to anti-diagonal form:*

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta, N+1}.$$

[†]Here d is a constant, and $d\eta(\partial_1, \partial_1)$ means the multiplication of the scalar $\eta(\partial_1, \partial_1)$ by this constant, not its exterior derivative.

In these coordinates the prepotential takes the form

$$F = \frac{1}{2}(t^1)^2(t^N) + \frac{1}{2}t^1 \sum_{\alpha=2}^{N-1} t^\alpha t^{N-\alpha+1} + f(t_2, \dots, t_N). \quad (1.2.11)$$

Furthermore, the sum

$$d_\alpha + d_{N-\alpha+1} = 2 - d \quad (1.2.12)$$

does not depend on α .

Proof. To begin with, we claim that if $\eta(\partial_1, \partial_1) = 0$, then we can define t^N in such a way that $\eta(\partial_1, \partial_N) = 1$ and retain that ∂_N is still an eigenvector of Q : Because η is non-degenerate, $\exists V \in \Gamma(T\mathcal{M}, \mathcal{M})$ such that $\eta(p_1, V) \neq 0$. Because Q is diagonalisable,

$$V = \sum_{i=2}^N v^i \partial_i,$$

where each ∂_i is an eigenvector for Q . But one can always multiply such eigenvectors by a complex scalar and obtain another eigenvector for the same eigenvalue. Thus by choosing appropriate scalar multiples of the ∂_i we obtain that V is an eigenvector for Q . We call it ∂_n . On the orthogonal complement of ∂_1 and ∂_n we can reduce the metric to anti-diagonal form by performing a \mathbb{C} -linear change of bases, again using this freedom to re-scale the eigenvectors. The formula (1.2.11) follows from (1.2.2).

The invariance of the sum (1.2.12) is equation $\mathcal{L}_E \eta = (2-d)\eta$ in this newly normalised basis of flat coordinates. \square

Example 4. Suppose $N = 2$. Then associativity is satisfied trivially because the algebras are unital. If $\eta_{11} = 0$ and $d \neq 1$, then the prepotential and Euler field can take the form

$$F(t_1, t_2) = \frac{1}{2}t_1^2 t_2 + f(t_2), \quad E = t_1 \frac{\partial}{\partial t_1} + (1-d)t_2 \frac{\partial}{\partial t_2}.$$

for some function $f(t_2)$. In examples, when the index of a coordinate field takes a specific value (like 1 or 2 here), we will identify raised and lowered indices, so $t_1 = t^1$, $t_2 = t^2$. If the index is not specified, the coordinate field with lowered index is achieved using the isomorphism generated by η . For example, $t_\alpha = \eta_{\alpha\epsilon} t^\epsilon$.

According to equation (1.2.9), this function must satisfy the homogeneity condition

$$\begin{aligned}
 E(F) &= (3-d)F \quad \text{modulo quadratic terms} \\
 \Rightarrow (d-1)t_2 f'(t_2) &= (3-d)f(t_2) + \alpha t_2^2 + \beta t_2 + \gamma \\
 t_2 f'(t_2) - \frac{3-d}{1-d} f(t_2) &= \tilde{\alpha} t_2^2 + \tilde{\beta} t_2 + \tilde{\gamma} \\
 \Rightarrow t_2^{-k} f(t_2) &= \int \left(\frac{\tilde{\alpha}}{t_2^{k-1}} + \frac{\tilde{\beta}}{t_2^k} + \frac{\tilde{\gamma}}{t_2^{k+1}} \right) dt_2; \quad k = \frac{3-d}{1-d}.
 \end{aligned}$$

Integrating this equation, one finds that

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + t_2^k, \quad k = \frac{3-d}{1-d}, \quad \text{for } d \neq -1, 1, 3, \quad (1.2.13)$$

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + t_2^2 \log t_2, \quad \text{for } d = -1, \quad (1.2.14)$$

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 - \log t_2, \quad \text{for } d = 3. \quad (1.2.15)$$

If $\eta_{11} = 0$ and $d = 1$, the Euler field takes the form

$$E = t_1 \frac{\partial}{\partial t_1} + r \frac{\partial}{\partial t_2},$$

for r constant. Homogeneity implies

$$r f'(t_2) = 2f,$$

which can be integrated to yield

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2 + e^{\frac{2}{r} t_2}, \quad \text{for } d = 1, r \neq 0, \quad (1.2.16)$$

$$F(t_1, t_2) = \frac{1}{2} t_1^2 t_2, \quad \text{for } d = 1, r = 0. \quad (1.2.17)$$

If $\eta_{11} \neq 0$, (as we saw above, this can only happen if $d = 0$) then we can take

$$F = \frac{1}{6} t_1^3 + \frac{1}{6} t_2^3 + f(t^2); \quad E = t_1 \frac{\partial}{\partial t_1} + t_2 \frac{\partial}{\partial t_2}.$$

Homogeneity gives that f is homogeneous of degree three up to a quadratic, and so

$$F(t_1, t_2) = \frac{1}{6} t_1^3 + \frac{1}{6} t_2^3 + r(t^2)^3 \quad \text{for } d = 0 \text{ and } r \text{ constant.} \quad (1.2.18)$$

This list (1.2.13) - (1.2.18) comprises a list of all two-dimensional semi-simple Frobenius manifolds. All the two dimensional Frobenius manifolds are linked, in a way that will be made precise, to the bi-Hamiltonian structures of certain hydrodynamic type partial

differential equations. In particular, we will see that the manifolds (1.2.14) and (1.2.15) describe the dispersionless limits of the bi-Hamiltonian structures of the Benney and Dym hierarchies. It may be shown [16] that the solutions (1.2.16) and (1.2.17) correspond to the dispersionless limits of the Toda and KdV hierarchies respectively.

When the dimension of the manifold is greater than two, one encounters more interesting nonlinear equations, but the problem of classification is much harder.

Example 5. Suppose $N = 3$ and let the Euler vector field and prepotential take the form

$$E = t^1 \frac{\partial}{\partial t^1} + \frac{1}{2} t^2 \frac{\partial}{\partial t^2}, \quad F(t^1, t^2, t^3) = \frac{1}{2} (t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{(t^2)^4}{16} \gamma(t^3). \quad (1.2.19)$$

The WDVV equations are equivalent to the nonlinear differential equation (known as the Chazy equation)

$$\gamma''' = 6\gamma\gamma'' - 9(\gamma')^2 \quad (1.2.20)$$

for the function $\gamma(t^3)$ (here ' denotes differentiation with respect to the variable t^3). It is interesting that this differential equation has an $SL(2, \mathbb{C})$ invariance, mapping solutions to solutions,

$$t^3 \mapsto \hat{t}^3 = \frac{at^3 + b}{ct^3 + d}, \quad \gamma(t^3) \mapsto \hat{\gamma}(\hat{t}^3) = (ct^3 + d)^2 \gamma(t^3) + 2c(ct^3 + d), \quad ad - bc = 1. \quad (1.2.21)$$

The metric in flat coordinates reads

$$\partial_1 \partial_\alpha \partial_\beta F = c_{1\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and the other non-zero elements of the multiplication are

$$\begin{aligned} c_{222} &= -\frac{3}{2} \gamma(t^3) t^2, & c_{223} &= -\frac{3}{4} (t^2)^2 \gamma'(t^3), \\ c_{233} &= -\frac{1}{4} (t^2)^3 \gamma''(t^3), & c_{333} &= -\frac{1}{16} (t^2)^4 \gamma'''(t^3). \end{aligned} \quad (1.2.22)$$

The hydrodynamic type system associated to this solution to WDVV is new. We will see later on that it inherits the modular symmetry present in the solution γ to the Chazy equation (2.1.2).

Next we introduce an important structure on a Frobenius manifold: the deformed Euclidean, or *Dubrovin* connection.

1.3 Flat Sections of The Dubrovin Connection

The deformed Euclidean, or Dubrovin connection provides an elegant re-packaging of the WDVV equations. Let (\mathcal{M}, η) be a Riemannian manifold, and ∇ the covariant derivative

for the corresponding Levi-Civita connection for η . Suppose further that we have a multiplication of vectors fields $\circ : \Gamma(T\mathcal{M}, \mathcal{M}) \times \Gamma(T\mathcal{M}, \mathcal{M}) \rightarrow \Gamma(T\mathcal{M}, \mathcal{M})$, and consider the 1-parameter family, $\tilde{\nabla}(\lambda)$, of connections defined by

$$\tilde{\nabla}(\lambda)_X Y = \nabla_X Y + \lambda X \circ Y. \quad (1.3.1)$$

Here λ is called the *spectral parameter*. Then we have

Theorem 2. *The connection $\tilde{\nabla}(\lambda)$ is torsion free if and only if \circ is commutative. It is flat identically in λ if and only if \circ is associative and*

$$\nabla_X(Y \circ Z) = \nabla_Y(X \circ Z), \quad X, Y, Z \in \Gamma(T\mathcal{M}). \quad (1.3.2)$$

Proof. The proof is by direct calculation and will be omitted. See, for example, [16], Lecture 3. \square

On a Frobenius manifold we have a commutative associative product. Further, writing the condition (1.3.2) in the flat coordinate system we have

$$(\partial_\varepsilon c_{\nu\kappa}^\sigma - \partial_\nu c_{\varepsilon\kappa}^\sigma) \partial_\sigma = 0.$$

Taking the inner product with ∂_μ we obtain the scalar-valued equation

$$\partial_\varepsilon c_{\mu\nu\kappa} - \partial_\nu c_{\mu\varepsilon\kappa} = 0,$$

which is exactly the potentiality axiom of a Frobenius manifold. So,

$$\{ \tilde{\nabla}(\lambda) \text{ is torsion free and flat } \forall \lambda \} \Leftrightarrow \{ \text{WDVV equations} \}.$$

In fact, one can also incorporate the quasihomogeneity property required for a Frobenius manifold: consider the extension of the connection $\tilde{\nabla}(\lambda)$ onto $T(\mathcal{M} \times \mathbb{C}^*)$ by the formulae

$$\begin{aligned} \tilde{\nabla}(\lambda)_{\partial_\lambda} X &= \partial_\lambda X + E \circ X + \frac{1}{\lambda} QX, \\ \tilde{\nabla}(\lambda)_{\partial_\lambda} \partial_\lambda &= 0, \\ \tilde{\nabla}(\lambda)_X \partial_\lambda &= 0, \end{aligned}$$

for $\partial_\lambda \in T\mathbb{C}^*$, and $X \in T\mathcal{M}$. Then the additional constraint $[\tilde{\nabla}(\lambda)_Y, \tilde{\nabla}(\lambda)_{\partial_\lambda}] = 0$, for $Y \in T\mathcal{M}$ gives

$$\{ \tilde{\nabla}(\lambda) \text{ is torsion free and flat } \forall \lambda \} \Leftrightarrow \{ \text{Frobenius manifold} \}.$$

Frobenius' integrability theorem states that the vanishing of the curvature of $\tilde{\nabla}(\lambda)$ im-

plies there exists N algebraically independent functions $h_\sigma(t^1, \dots, t^N; \lambda)$, such that

$$\tilde{\nabla}(\lambda)dh_\sigma(\mathbf{t}; \lambda) = 0, \quad \text{for } \sigma = 1, \dots, N. \quad (1.3.3)$$

To put it another way, the functions $\{h_\sigma : \sigma = 1, \dots, N\}$ comprise a system of flat coordinates for the connection $\tilde{\nabla}(\lambda)$. Writing equation (1.3.3) in the flat coordinate system for the metric η , this reads

$$\frac{\partial^2 h_\sigma}{\partial t^\varepsilon \partial t^\kappa} = \lambda c_{\varepsilon\kappa}^\mu \frac{\partial h_\sigma}{\partial t^\mu}. \quad (1.3.4)$$

Givental originally coined the term ‘*quantum differential equation*’ for equation (1.3.4) for examples of Frobenius manifolds coming from quantum cohomology. In this case the structure functions of the Frobenius algebras are themselves Gromov-Witten invariants of the symplectic manifold in question. For example, the solution (1.2.16) generates the genus zero Gromov-Witten invariants of \mathbb{P}^1 .

We will assume, following [16], that the solutions of this system are normalised in the following way:

$$h_\sigma(\mathbf{t}, 0) = t_\sigma := \eta_{\sigma\alpha} t^\alpha, \quad (1.3.5)$$

$$\langle \nabla h_\mu(\mathbf{t}, \lambda), \nabla h_\nu(\mathbf{t}; -\lambda) \rangle = \eta_{\mu\nu}, \quad (1.3.6)$$

$$\partial_1 h_\sigma(\mathbf{t}; \lambda) = \lambda h_\sigma(\mathbf{t}; z) + \eta_{1\sigma}. \quad (1.3.7)$$

The first normalisation (1.3.5) is motivated by setting $\lambda = 0$: we recover the original Levi-Civita connection for η , and so view the functions h_σ as perturbations of the original flat coordinate system. Hence one may seek power series solutions of the form

$$h_\sigma(\mathbf{t}; \lambda) = t_\sigma + \sum_{n \geq 1} \lambda^n h_{n,\sigma}(\mathbf{t}).$$

With this ansatz, we get the recursion relation

$$\frac{\partial^2 h_{n,\sigma}}{\partial t^\varepsilon \partial t^\kappa} = c_{\varepsilon\kappa}^\nu \frac{\partial h_{n-1,\sigma}}{\partial t^\nu}, \quad h_{0,\sigma} = t_\sigma. \quad (1.3.8)$$

The unity field gives the simplified recurrence relation

$$\frac{\partial^2 h_{n,\sigma}}{\partial t^1 \partial t^\varepsilon} = \frac{\partial h_{n-1,\sigma}}{\partial t^\varepsilon},$$

which is (1.3.7) in terms of the coefficients $h_{n,\sigma}$.

Remark. In physical literature, the coefficients $h_{n,\alpha}$ appearing in the expansion of these ‘*deformed flat coordinates*’ are known as *gravitational descendants*. The recursion relation (1.3.8) first appeared in the paper [67] of Witten who wrote it down for the trivial

Frobenius manifold,

$$F = \frac{1}{6}t^3; \quad E = t \frac{\partial}{\partial t}$$

Note that because for two solutions h_μ, h_ν of (1.3.4) we have $\nabla \langle \nabla h_\mu(t, \lambda), \nabla h_\nu(t; -\lambda) \rangle = 0$, the assignment (1.3.6) is the canonical choice of a covariant constant, and defines a bilinear pairing on the space of solutions.

Define the matrices $R_r, r \geq 0$ by

$$\mathcal{L}_E h_{n,\sigma}(\mathbf{t}) = (d_{N-\sigma+1} + n)h_{n,\sigma}(\mathbf{t}) + \sum_{r=1}^N \sum_{v=1}^N h_{n-r,v}(\mathbf{t})(R_r)_\sigma^v. \quad (1.3.9)$$

The matrices R_r are constant and are characterised by the properties

$$(R_r)_\sigma^v \neq 0 \quad \text{only if} \quad \mu_v - \mu_\sigma = r, \quad \eta_{\sigma\gamma}(R_r)_\beta^\gamma + (-1)^r \eta_{\beta\gamma}(R_r)_\sigma^\gamma = 0,$$

where $\mu_\alpha := 1 - d_\alpha - \frac{d}{2}$. The equation (1.3.9) fixes a choice of basis of deformed flat coordinates uniquely. This is a nontrivial fact due to Dubrovin & Zhang. For a more comprehensive exposition of the definition of the matrices, and fixing the basis of solutions to (1.3.4) see, for example, [19]. Note that the number of non-zero matrices R_r is bounded above by the dimension of the Frobenius manifold. The motivation for equation (1.3.9) is that ideally one would like to simply stipulate

$$\mathcal{L}_E h_{n,\sigma} = (d_{N-\sigma+1} + n)h_{n,\sigma}, \quad (1.3.10)$$

but this does not fix the deformed flat coordinates uniquely in all cases: if $d_{N-\lambda+1} + n - r = d_{N-\sigma+1} + n$, i.e. the weights $d_{N-\lambda+1}$ and $d_{N-\sigma+1}$ differ by an integer r , then

$$\mathcal{L}_E h_{n,\sigma} = \mathcal{L}_E h_{n-r,\lambda}.$$

Thus (1.3.9) absorbs this ambiguity. The latter property of the matrices is to ensure consistency with the normalisation (1.3.6). Frobenius manifolds for which any of the matrices R_r are non-zero are called *resonant*. Otherwise the Frobenius manifold is said to be *non-resonant*. As we will see in later sections the solutions to these equations will be extremely important to us.

Example 6. [19] For a Frobenius manifold whose flat coordinate system is normalised as in Lemma 1, we have

$$F(t) = \frac{1}{2} \left\{ \langle \nabla h_{1,\alpha}, \nabla h_{1,1} \rangle \eta^{\alpha\beta} \langle \nabla h_{0,\beta}, \nabla h_{1,1} \rangle - \langle \nabla h_{1,1}, \nabla h_{2,1} \rangle - \langle \nabla h_{3,1}, \nabla h_{0,1} \rangle \right\}.$$

Using (1.3.9) we find, upon comparison with (1.2.9), that

$$\begin{aligned} A_{\alpha\beta} &= (R_1)_\alpha^\varepsilon \eta_{\varepsilon\beta}, \\ B_\alpha &= (R_2)_\alpha^\varepsilon \eta_{\varepsilon 1}, \\ C &= -\frac{1}{2}(R_3)_1^\varepsilon \eta_{\varepsilon 1}. \end{aligned}$$

Example 7. For the trivial Frobenius manifold,

$$F(t) = \frac{t^3}{6}, \quad E = t \frac{\partial}{\partial t},$$

the quantum differential equation reads

$$\frac{\partial h(t; z)}{\partial t} = zh(t; z).$$

Choosing

$$h(t; z) = \frac{1}{z} e^{tz},$$

we satisfy the normalisation $h_0(t) = t$. Because the Frobenius manifold is one dimensional, all the constants $R_r = 0$, $r \geq 0$. As a power series,

$$h(t; z) = \sum_{k \geq 0} z^k \frac{t^{k+1}}{(k+1)!}. \quad (1.3.11)$$

We will come to recognise the deformed flat coordinates $h_n(t)$ as the dispersionless limits of conservation laws for the KdV hierarchy.

Example 8. [24, 65] For the Frobenius manifold defined by the data

$$\begin{aligned} F(t^1, t^2) &= \frac{1}{2}(t^1)^2 t^2 + (t^2)^2 \log t^2, \\ E &= t^1 \frac{\partial}{\partial t^1} + 2t^2 \frac{\partial}{\partial t^2}, \end{aligned}$$

the quantum differential equation (1.3.4) decouples (this is true for any two dimensional Frobenius manifold):

$$\begin{aligned} \frac{\partial h_\sigma(\mathbf{t}; \lambda)}{\partial t^1 \partial t^\beta} &= \lambda \delta_\beta^\varepsilon \frac{\partial h_\sigma(\mathbf{t}; \lambda)}{\partial t_\varepsilon} = \lambda \frac{\partial h_\sigma(\mathbf{t}; \lambda)}{\partial t^\beta} \\ \Rightarrow \frac{\partial}{\partial t^\beta} \left(\frac{\partial h_\sigma(\mathbf{t}; \lambda)}{\partial t^1} - \lambda h_\sigma(\mathbf{t}; \lambda) \right) &= 0, \quad \text{for } \beta = 1, 2 \\ \Rightarrow \frac{\partial h_\sigma(\mathbf{t}; \lambda)}{\partial t^1} &= \lambda h_\sigma(\mathbf{t}; \lambda) + G(\lambda) \end{aligned}$$

for some function $G(\lambda)$. This equation can be solved using an integrating factor to give

$$h_\sigma(\mathbf{t}; \lambda) = -\frac{1}{\lambda}G(\lambda) + e^{t^1\lambda}H(t^2; \lambda). \quad (1.3.12)$$

On the other hand, we also have

$$\frac{\partial^2 h_\sigma(\mathbf{t}; \lambda)}{\partial t^2 \partial t^2} = \lambda c_{22}^\varepsilon \frac{\partial h_\sigma(\mathbf{t}; \lambda)}{\partial t^\varepsilon} = \lambda c_{22}^1 \frac{\partial h_\sigma(\mathbf{t}; \lambda)}{\partial t^1},$$

since we assume $\eta_{\alpha\beta} = c_{1\alpha\beta} = \delta_{\alpha+\beta, N+1}$. Upon substitution of the expression (1.3.12) we obtain the second order ordinary differential equation for $H(t^2; \lambda)$:

$$\frac{\partial^2 H(t^2; \lambda)}{\partial t^2 \partial t^2} = \frac{\lambda^2}{t^2} H(t^2; \lambda),$$

which is a slightly re-scaled version of the well-known Bessel equation. The general solution takes the form

$$H(t^2; \lambda) = \alpha_0 \lambda \sqrt{t^2} I_1(2\lambda \sqrt{t^2}) + \alpha_1 2\lambda \sqrt{t^2} K_1(2\lambda \sqrt{t^2})$$

for scalars α_0 and α_1 , and I_1 , K_1 denote the modified Bessel functions of the first and second kind [68]:

$$I_1(z) = \sum_{k \geq 0} \frac{1}{k!(k+1)!} \left(\frac{z}{2}\right)^{2k+1}, \quad (1.3.13)$$

$$K_1(z) = \left(\gamma + \log\left(\frac{z}{2}\right)\right) I_1(z) - \sum_{k \geq 0} \frac{1}{k!(k+1)!} \left(\frac{z}{2}\right)^{2k+1} (H_k + H_{k+1}). \quad (1.3.14)$$

γ is the Euler–Mascheroni constant and H_k is the k^{th} Harmonic number,

$$H_k = \sum_{r=1}^k \frac{1}{r}.$$

Choosing

$$\begin{aligned} h_1(\mathbf{t}; \lambda) &= e^{t^1\lambda} \left\{ \sqrt{t^2} K_1(2\lambda \sqrt{t^2}) - (\gamma + \log \lambda) \sqrt{t^2} I_1(2\lambda \sqrt{t^2}) + \frac{1}{\lambda} \right\} - \frac{1}{\lambda}, \\ h_2(\mathbf{t}; \lambda) &= e^{t^1\lambda} \left\{ \frac{\sqrt{t^2}}{\lambda} I_1(2\lambda \sqrt{t^2}) \right\}, \end{aligned}$$

we satisfy the normalisation conditions. As power series these read:

$$h_1(\mathbf{t}; \lambda) = e^{t^1 \lambda} \left\{ \frac{1}{\lambda} + \sum_{k \geq 0} \frac{1}{k!(k+1)!} (t^2)^{k+1} \lambda^{2n} (\log t^2 - H_k - H_{k+1}) \right\} + \frac{1}{\lambda}, \quad (1.3.15)$$

$$h_2(\mathbf{t}; \lambda) = e^{t^1 \lambda} \left\{ \sum_{k \geq 0} \frac{1}{k!(k+1)!} (t^2)^{k+1} \lambda^{2n} \right\}. \quad (1.3.16)$$

Note that because F satisfies the homogeneity condition $\mathcal{L}_E(F) = 4F + 2(t^2)^2$, we have

$$R_1 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}; \quad R_i = 0, \quad \text{for } i > 1.$$

The coefficients appearing here in the expansions of the deformed flat coordinates (1.3.15), (1.3.16) turn out to be dispersionless limits of conservation laws of the Benney hierarchy.

When the dimension of the manifold exceeds two it becomes more difficult to find a closed form expression for the deformed flat coordinates. But we can still solve the recursion:

Example 9 (Continued from Example 5, page 12). For the solution (1.2.19), we have the recursion relations

$$\frac{\partial^2 h_\sigma^{(n)}}{\partial t^\mu \partial t^\nu} = c_{\mu\nu}^\varepsilon \frac{\partial h_\sigma^{(n-1)}}{\partial t^\varepsilon}, \quad \text{for } \mu, \nu, \sigma = 1, 2, 3, \quad n \geq 0,$$

subject to $h_\sigma^{(0)} = t_\sigma$. Solving this equation recursively gives the solutions

$$\begin{aligned} h_1(t; \lambda) &= t^3 + \lambda \left\{ t^1 t^3 + \frac{1}{2} (t^2)^2 \right\} \\ &\quad + \lambda^2 \left\{ -\frac{1}{16} t^3 (t^2)^4 \gamma'(t^3) - \frac{1}{8} (t^2)^4 \gamma(t^3) + \frac{1}{2} t^1 (t^2)^2 + \frac{1}{2} (t^1)^2 t^3 \right\} + \mathcal{O}(\lambda^3), \\ h_2(t; \lambda) &= t^2 + \lambda \left\{ t^1 t^2 - \frac{1}{4} (t^2)^3 \gamma(t^3) \right\} \\ &\quad + \lambda^2 \left\{ -\frac{1}{20} (t^2)^5 \gamma'(t^3) + \frac{9}{160} (t^2)^5 \gamma(t^3)^2 - \frac{1}{4} t^1 (t^2)^3 \gamma(t^3) + \frac{1}{2} (t^1)^2 t^2 \right\} + \mathcal{O}(\lambda^3), \\ h_3(t; \lambda) &= t^1 + \lambda \left\{ \frac{1}{2} (t^1)^2 - \frac{1}{16} (t^2)^4 \gamma'(t^3) \right\} \\ &\quad + \lambda^2 \left\{ -\frac{1}{480} (t^2)^6 \gamma''(t^3) + \frac{1}{80} (t^2)^6 \gamma(t^3) \gamma'(t^3) - \frac{1}{16} t^1 (t^2)^4 \gamma'(t^3) + \frac{1}{6} (t^1)^3 \right\} + \mathcal{O}(\lambda^3). \end{aligned}$$

We will use these deformed flat coordinates to generate commuting functionals for a new integrable hierarchy. One can verify equations (1.3.5) - (1.3.7) explicitly up to $\mathcal{O}(\lambda^3)$. This solution to WDVV satisfies the homogeneity condition $\mathcal{L}_E F = 2F$, and so $R_i = 0$, $i \geq 0$.

1.4 Semi-Simple Frobenius Manifolds

Throughout this thesis, we will restrict ourselves to the case where the Frobenius algebra structure on the tangent spaces to our manifold is generically semi-simple. This leads to the construction of another coordinate system that capitalises on this simple algebraic structure. They are known in the literature as *canonical coordinates*. As we will see this new coordinate system can be difficult to construct explicitly (it will involve finding the roots of a degree N polynomial, where $N = \dim(\mathcal{M})$), but they can be an extremely useful tool for proving results on semi-simple Frobenius manifolds in general.

Definition 4. Let (A, \circ, e) be a unital algebra of dimension N (as a vector space). We say that A is semi-simple if it decomposes as the direct sum of N one-dimensional algebras. That is, there exists a basis of idempotents $\{u_1, \dots, u_N\}$ such that

$$u_i \circ u_j = \delta_{ij} u_j.$$

We will call a point $t \in \mathcal{M}$ semi-simple if the the Frobenius algebra on the tangent space $T_t \mathcal{M}$ is semi-simple. If the tangent algebras are generically semi-simple, then we say that the Frobenius manifold is semi-simple.

Lemma 2. [16] Let $(\mathcal{M}, \eta, \circ, e, E)$ comprise a Frobenius manifold. In a neighbourhood of a semi-simple point, canonical coordinates exists such that

$$\frac{\partial}{\partial u_i} \circ \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_j}. \quad (1.4.1)$$

Proof. In a neighbourhood of a semi-simple point t , a basis of vector fields $\{\xi_1, \dots, \xi_N\}$ exists such that

$$\xi_i \circ \xi_j = \delta_{ij} \xi_j.$$

To show that this is a coordinate basis, i.e. $\xi_i = \partial/\partial u_i$ for some coordinates system $\{u_1, \dots, u_N\}$, we must show that the vector fields commute pairwise. To this end consider the obstructions to commutativity,

$$\sum_{k=1}^N f_{ij}^k \xi_k = [\xi_i, \xi_j].$$

Writing the zero curvature condition of $\tilde{\nabla}(\lambda)$ in the basis of idempotents, we have

$$\Gamma_{kj}^l \delta_i^l + \Gamma_{ki}^l \delta_{kj}^l - \Gamma_{ki}^l \delta_j^l - \Gamma_{kj}^l \delta_{ki}^l = f_{ij}^l \delta_k^l,$$

where Γ_{jk}^i are the Christoffel symbols of the metric η . Choosing $l = k$ gives $f_{ij}^k = 0$. \square

Latin indices will be used when working in canonical coordinates, and the summation

convention will be suspended unless otherwise stated. Let us investigate how some of the main tensors on a Frobenius manifold look.

- The unity field takes the form

$$e = \sum_{i=1}^N \frac{\partial}{\partial u_i}. \quad (1.4.2)$$

This follows from

$$\left(\sum_{i=1}^N \frac{\partial}{\partial u_i} \right) \circ \frac{\partial}{\partial u_j} = \frac{\partial}{\partial u_j} \circ \frac{\partial}{\partial u_j} = \frac{\partial}{\partial u_j}.$$

- The Euler field takes the form

$$E = \sum_{i=1}^N u_i \frac{\partial}{\partial u_i}. \quad (1.4.3)$$

To see this, note that the multiplication is invariant under the re-scalings $u \mapsto ku$, for constant k . Then the axiom $\mathcal{L}_E \circ = \circ$ is equivalent to E being the vector field (1.4.3) generating this one-parameter family of diffeomorphisms: $f(u) = ku$ satisfies $E(f) = ku$, i.e. $u \mapsto ku$ is an integral curve for E .

- The metric is diagonal,

$$\eta \left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right) = \eta_{ii}(u) \delta_{ij},$$

for some non-zero functions $\eta_{11}(u), \dots, \eta_{NN}(u)$:

$$\eta \left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right) = \eta \left(e \circ \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right) = \eta \left(e, \frac{\partial}{\partial u_i} \circ \frac{\partial}{\partial u_j} \right) = \delta_{ij} \eta \left(e, \frac{\partial}{\partial u_j} \right).$$

So

$$\eta_{ii}(u) = \eta \left(e, \frac{\partial}{\partial u_i} \right). \quad (1.4.4)$$

- The tensor c takes the form

$$c \left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k} \right) = \eta_{ii}(u) \delta_{ij} \delta_{jk},$$

which is immediate from its definition.

From a geometer's point of view, the canonical coordinates comprise a system of *curvilinear orthogonal* coordinates. Because the metric is diagonal, most of the Christoffel symbols, and therefore components of the Riemann curvature tensor, vanish. Let us compute how the Christoffel symbols for a diagonal metric look. Recall that

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{s=1}^N \eta^{ks} (\partial_i \eta_{js} + \partial_j \eta_{is} - \partial_s \eta_{ij}),$$

and so for a diagonal metric we have

$$\Gamma_{ij}^k = \frac{1}{2} \eta^{kk} (\partial_i \eta_{jk} + \partial_j \eta_{ik} - \partial_k \eta_{ij}) \quad (\text{no sum}).$$

It is clear then that for i, j, k all distinct, $\Gamma_{ij}^k = 0$. Suppose $i = j = k$. Then

$$\Gamma_{kk}^k = \frac{1}{2} \eta^{kk} \partial_k \eta_{kk}.$$

It is convenient to introduce the *rotation coefficients*, $\beta_{ij}(u)$, which are defined by

$$\beta_{ij} = \frac{\partial_i \sqrt{\eta_{jj}(u)}}{\sqrt{\eta_{ii}(u)}}. \quad (1.4.5)$$

In general, $\beta_{ij} \neq \beta_{ji}$. In terms of the rotation coefficients, we have

$$\Gamma_{kk}^k = \beta_{kk}.$$

If $i \neq k$, we have

$$\begin{aligned} \Gamma_{ii}^k &= -\frac{1}{2} \eta^{kk} \partial_k \eta_{ii} = -\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{kk}}} \beta_{ki}; \\ \Gamma_{ki}^k &= \frac{1}{2} \eta^{kk} \partial_i \eta_{kk} = \frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{kk}}} \beta_{ik}. \end{aligned}$$

Proposition 1. [17] *On a Frobenius manifold, the rotation coefficients are symmetric,*

$$\beta_{ij} = \beta_{ji},$$

and so the metric is Egoroff, with potential $t_1(u)$,

$$\eta_{ii}(u) = \partial_i t_1(u).$$

Furthermore, the rotation coefficients β_{ij} are quasi-homogeneous with weight -1:

$$E(\beta_{ij}) = -\beta_{ij}.$$

Conversely, suppose we have a metric that is Egoroff with respect to a coordinate system $\{u_1, \dots, u_N\}$, and whose rotation coefficients satisfy the non-linear system of differential

equations

$$\partial_i \beta_{jk} = \beta_{ik} \beta_{kj}, \quad (1.4.6)$$

$$\sum_i \partial_i \beta_{jk} = 0, \quad (1.4.7)$$

$$\sum_i u_i \partial_i \beta_{jk} = -\beta_{jk}. \quad (1.4.8)$$

Then the structure of a semi-simple Frobenius manifold exists with canonical coordinates $\{u_1, \dots, u_N\}$.

Remark. The system of equations (1.4.6) - (1.4.8) is known as the Darboux-Egoroff system. Choosing the canonical coordinates has simplified the algebraic structure, and so the WDVV equations; instead the non-linearity manifests itself as the zero-curvature of the metric η .

Proof. On a Frobenius manifold, the unity field is covariantly constant. Then it follows from the expression for e in canonical coordinates (1.4.2) that

$$\sum_{s=1}^N \Gamma_{ks}^r = 0 \quad \text{for } r, k = 1, \dots, N.$$

In particular, if $r \neq k$,

$$\Gamma_{rk}^r + \Gamma_{kk}^r = 0 \Rightarrow \beta_{rk} = \beta_{kr} \Rightarrow \partial_r \eta_{kk}(u) = \partial_k \eta_{rr}(u).$$

So the one-form $\eta(e, -)$ is closed. Recall that we have arranged the flat coordinates in such a way that the metric is anti-diagonal, and so

$$\eta\left(\frac{\partial}{\partial t^1}, -\right) = dt_1.$$

Secondly, the Euler field is covariantly linear,

$$\begin{aligned} \nabla_j \nabla_i E^k &= \nabla_j \nabla_i u_k = 0, \quad \text{for } i, j, k = 1, \dots, N \\ \Leftrightarrow \sum_{p=1}^N \partial_j \Gamma_{ip}^k u_p + \Gamma_{ij}^k + \sum_{s,p=1}^N u_p \left(\Gamma_{sj}^k \Gamma_{ip}^s - \Gamma_{ij}^s \Gamma_{sp}^k \right) &= 0. \end{aligned} \quad (1.4.9)$$

Now use the fact that the curvature of the metric η is zero,

$$R_{ijp}^k = 0 \quad \Leftrightarrow \quad \sum_{s=1}^N \Gamma_{sj}^k \Gamma_{ip}^s - \Gamma_{ij}^s \Gamma_{sp}^k = -\partial_j \Gamma_{ip}^k + \partial_p \Gamma_{ij}^k,$$

to rewrite this expression (1.4.9) as

$$\begin{aligned} \sum_{p=1}^N u_p \left(\partial_j \Gamma_{ip}^k + \partial_p \Gamma_{ij}^k - \partial_j \Gamma_{ip}^k \right) + \Gamma_{ij}^k &= 0 \\ \Leftrightarrow \sum_{p=1}^N u_p \partial_p \Gamma_{ij}^k + \Gamma_{ij}^k &= 0 \\ \Leftrightarrow \sum_{p=1}^N u_p \partial_p \beta_{ij} &= -\beta_{ij}. \end{aligned}$$

To show the converse, one uses standard formulas of Riemannian geometry to show that the axioms of a Frobenius manifold in canonical coordinates are satisfied. The Riemann curvature tensor in the canonical coordinates $\{u_i : i = 1, \dots, N\}$ is given by

$$R_{kl}^{ij} = \sum_{s,p=1}^N g^{is} \left(\partial_k \Gamma_{sl}^j - \partial_l \Gamma_{sk}^j + \Gamma_{pk}^i \Gamma_{sl}^p - \Gamma_{pl}^j \Gamma_{sk}^p \right). \quad (1.4.10)$$

It is clear that for a diagonal metric, $R_{kl}^{ij} = 0$ for i, j, k, l all distinct. Further, standard skew-symmetries of the curvature tensor give

$$R_{il}^{ij} = -R_{li}^{ij} = R_{il}^{ji} = -R_{il}^{ji}.$$

By definition $R_{kk}^{ij} = 0$, $\forall i, j, k$. We also have

$$R_{kl}^{ii} = \eta^{ii} (\partial_k \partial_l \sqrt{\eta_{ii}} - \partial_l \partial_k \sqrt{\eta_{ii}}) = 0 \quad \Leftrightarrow \quad [\partial_l, \partial_k] = 0,$$

which is true irrespective of the values of l and k . Owing to the standard symmetries of the Riemann tensor, we are left with the following cases to analyse:

- i) R_{il}^{ij} for i, j and l distinct;
- ii) R_{ij}^{ij} for i and j distinct.

Case i) We have

$$R_{il}^{ij} = \sum_{s,p=1}^N \eta^{is} \left(\partial_i \Gamma_{sl,i}^j - \partial_l \Gamma_{si}^j + \Gamma_{pi}^j \Gamma_{sl}^p - \Gamma_{pl}^j \Gamma_{si}^p \right) = \underbrace{\frac{1}{\eta_{ii}} \left(\partial_i \Gamma_{il}^j - \partial_l \Gamma_{ii}^j \right)}_A - \underbrace{\frac{1}{\eta_{ii}} \sum_{p=1}^N \left(\Gamma_{pi}^j \Gamma_{il}^p - \Gamma_{pl}^j \Gamma_{ii}^p \right)}_B.$$

In this case, i, j and l are distinct and so $\Gamma_{jl}^i = 0$, and we obtain for the first term

$$\begin{aligned} A &= \frac{1}{\eta_{ii}} \partial_l \left(\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ji} \right) = \frac{1}{\eta_{ii}} \left(\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \frac{\partial_l \sqrt{\eta_{ii}}}{\sqrt{\eta_{ll}}} - \frac{\sqrt{\eta_{ii}}}{\eta_{jj}} \partial_l \sqrt{\eta_{jj}} \right) + \frac{1}{\sqrt{\eta_{ii}} \sqrt{\eta_{jj}}} \beta_{ji,l} \\ &= \frac{\sqrt{\eta_{ii}}}{\eta_{ii} \sqrt{\eta_{jj}}} \beta_{li} \beta_{ji} - \frac{\sqrt{\eta_{ii}}}{\eta_{jj} \sqrt{\eta_{ii}}} + \frac{1}{\sqrt{\eta_{ii}} \sqrt{\eta_{jj}}} \beta_{ji,l}. \end{aligned}$$

For the second term,

$$\begin{aligned}
B &= \frac{1}{\eta_{ii}} \sum_{p=1}^N \left(\Gamma_{pi}^j \Gamma_{il}^p - \Gamma_{pl}^j \Gamma_{ii}^p \right) = \frac{1}{\eta_{ii}} \left(\Gamma_{ii}^i \Gamma_{il}^i + \Gamma_{li}^j \Gamma_{il}^l - \Gamma_{jl}^j \Gamma_{ii}^j - \Gamma_{ll}^j \Gamma_{ii}^l \right) \\
&= \frac{1}{\eta_{ii}} \left(\left\{ -\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ji} \right\} \left\{ \frac{\sqrt{\eta_{il}}}{\sqrt{\eta_{ii}}} \beta_{li} \right\} - \left\{ \frac{\sqrt{\eta_{il}}}{\sqrt{\eta_{jj}}} \beta_{li} \right\} \left\{ -\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ji} \right\} - \left\{ -\frac{\sqrt{\eta_{il}}}{\sqrt{\eta_{jj}}} \beta_{jl} \right\} \left\{ -\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{il}}} \beta_{li} \right\} \right) \\
&= -\frac{\sqrt{\eta_{il}}}{\eta_{ii}\sqrt{\eta_{jj}}} \beta_{li} \beta_{ji} + \frac{\sqrt{\eta_{il}}}{\eta_{jj}\sqrt{\eta_{ii}}} \beta_{lj} \beta_{ji} - \frac{1}{\sqrt{\eta_{ii}}\sqrt{\eta_{jj}}} \beta_{jl} \beta_{li}.
\end{aligned}$$

Therefore,

$$R_{il}^{ij} = \frac{1}{\sqrt{\eta_{ii}}\sqrt{\eta_{jj}}} (\partial_i \beta_{ji} - \beta_{jl} \beta_{li}). \quad (1.4.11)$$

Case ii) R_{ij}^{ij} for $i \neq j$. We have

$$\sum_{s,p=1}^N \eta^{is} \left(\partial_i \Gamma_{sj}^j - \partial_j \Gamma_{si}^j + \Gamma_{pi}^j \Gamma_{sj}^p - \Gamma_{pj}^j \Gamma_{si}^p \right) = \underbrace{\frac{1}{\eta_{ii}} (\partial_i \Gamma_{ij}^j - \partial_j \Gamma_{ii}^j)}_C + \underbrace{\frac{1}{\eta_{ii}} \sum_p (\Gamma_{pi}^j \Gamma_{ij}^p - \Gamma_{pj}^j \Gamma_{ii}^p)}_D.$$

The first term reads

$$\begin{aligned}
C &= \frac{1}{\eta_{ii}} \left\{ \partial_i \left(\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ij} \right) + \partial_j \left(\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ji} \right) \right\} \\
&= \frac{1}{\eta_{ii}} \left\{ \frac{\partial_i \sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ij} - \frac{\sqrt{\eta_{ii}} \partial_i \sqrt{\eta_{jj}}}{\eta_{jj}} \beta_{ij} + \frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \partial_i \beta_{ij} + \frac{\partial_j \sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ji} - \frac{\sqrt{\eta_{ii}} \partial_j \sqrt{\eta_{jj}}}{\eta_{jj}} \beta_{ji} + \frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \partial_j \beta_{ji} \right\} \\
&= \frac{1}{\sqrt{\eta_{ii}}\sqrt{\eta_{jj}}} \left\{ \partial_i \beta_{ij} + \partial_j \beta_{ji} + \beta_{ii} \beta_{ij} - \beta_{jj} \beta_{ji} - \frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \left(\frac{\partial_i \sqrt{\eta_{jj}}}{\sqrt{\eta_{ii}}} \right) \beta_{ij} + \left(\frac{\partial_j \sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \right) \frac{\sqrt{\eta_{jj}}}{\sqrt{\eta_{ii}}} \beta_{ji} \right\} \\
&= \frac{1}{\sqrt{\eta_{ii}}\sqrt{\eta_{jj}}} \left\{ \partial_i \beta_{ij} + \partial_j \beta_{ji} + \beta_{ii} \beta_{ij} - \beta_{jj} \beta_{ji} - \frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ij}^2 + \frac{\sqrt{\eta_{jj}}}{\sqrt{\eta_{ii}}} \beta_{ji}^2 \right\},
\end{aligned}$$

and the second

$$\begin{aligned}
D &= \frac{1}{\eta_{ii}} \sum_p \left(\Gamma_{pi}^j \Gamma_{ij}^p - \Gamma_{pj}^j \Gamma_{ii}^p \right) = \frac{1}{\eta_{ii}} \sum_p \left(\Gamma_{ii}^j \Gamma_{ij}^i + \Gamma_{ji}^j \Gamma_{ij}^j - \Gamma_{ij}^j \Gamma_{ii}^i - \Gamma_{jj}^j \Gamma_{ii}^j - \sum_{p \neq i,j} \Gamma_{jp}^j \Gamma_{ii}^p \right) \\
&= \frac{1}{\eta_{ii}} \left\{ \left(-\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ji} \right) \left(\frac{\sqrt{\eta_{jj}}}{\sqrt{\eta_{ii}}} \beta_{ji} \right) + \left(\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ij} \right) \left(\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ij} \right) - \left(\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ij} \right) (\beta_{ii}) \right. \\
&\quad \left. - (\beta_{jj}) \left(-\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ji} \right) + \sum_{p \neq i,j} \left(\frac{\sqrt{\eta_{pp}}}{\sqrt{\eta_{jj}}} \beta_{pj} \right) \left(\frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{pp}}} \beta_{pi} \right) \right\} \\
&= \frac{1}{\sqrt{\eta_{ii}}\sqrt{\eta_{jj}}} \left\{ \sum_{p \neq i,j} \beta_{pj} \beta_{pi} - \frac{\sqrt{\eta_{jj}}}{\sqrt{\eta_{ii}}} \beta_{ji}^2 + \frac{\sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}} \beta_{ij}^2 - \beta_{ij} \beta_{ii} + \beta_{jj} \beta_{ji} \right\}.
\end{aligned}$$

Therefore,

$$R_{ij}^{ij} = \frac{1}{\sqrt{\eta_{ii}}\sqrt{\eta_{jj}}} \left(\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{p \neq i,j} \beta_{pj} \beta_{pi} \right). \quad (1.4.12)$$

We arrive at the statement

$$R = 0 \Leftrightarrow \begin{cases} \partial_l \beta_{ji} - \beta_{jl} \beta_{li} = 0 \\ \partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{p \neq i,j} \beta_{pj} \beta_{pi} = 0 \end{cases}. \quad (1.4.13)$$

This system (1.4.13) for the zero curvature of a diagonal metric is known as the Darboux system. Adding the assumption that the metric is Egoroff implies that the rotation coefficients are symmetric:

$$\beta_{ij} = \frac{1}{2} \frac{\partial_i \partial_j \Phi}{\sqrt{\partial_i \Phi \partial_j \Phi}}. \quad (1.4.14)$$

Using the symmetry of the rotation coefficients, together with the former of the two equations, we have

$$(1.4.12) \Rightarrow \partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{p \neq i,j} \beta_{pj} \beta_{pi} = 0 \xLeftrightarrow{\beta_{ij} = \beta_{ji}} \sum_{p=1}^N \partial_p \beta_{ij} = 0. \quad (1.4.15)$$

Hence the Egoroff property of the metric gives a refinement of (1.4.13):

$$R = 0 \Leftrightarrow \begin{cases} \partial_l \beta_{ji} - \beta_{jl} \beta_{li} = 0 \\ \sum_{p=1}^N \partial_p \beta_{ij} = 0 \end{cases}. \quad (1.4.16)$$

We have already seen that the covariant linearity of the Euler field is equivalent to (1.4.8), and so it only remains to show that the 4-tensor $\nabla_k \delta_{ij} \delta_{jl}$ is totally symmetric. This follows from the the symmetry

$$\Gamma_{ij}^i = -\Gamma_{jj}^i$$

of the Christoffel symbols. □

1.4.1 The Intersection Form

On a Frobenius manifold, there exists a second metric defined on the cotangent bundle,

$$g(\omega_1, \omega_2) = \iota_E(\omega_1 \circ \omega_2), \quad \text{for } \omega_1, \omega_2 \in \Gamma(T^* \mathcal{M}, \mathcal{M}). \quad (1.4.17)$$

In flat coordinates,

$$g(dt^\alpha, dt^\beta) = E^\varepsilon c_\varepsilon^{\alpha\beta},$$

where the original metric η is used to provide the isomorphism between $T \mathcal{M}$ and $T^* \mathcal{M}$. The second metric g is called the *intersection form* of the Frobenius manifold.

Example 10. For the Frobenius manifold of Example 5 presented on page 12, the intersection form reads

$$g^{\alpha\beta} = \begin{pmatrix} -\frac{1}{8}(t_2)^4\gamma''(t_3) & -\frac{3}{8}(t_2)^3\gamma'(t_3) & t_1 \\ -\frac{3}{8}(t_2)^3\gamma'(t_3) & t_1 - \frac{3}{4}\gamma(t_3)(t_2)^2 & \frac{1}{2}t_2 \\ t_1 & \frac{1}{2}t_2 & 0 \end{pmatrix}.$$

Note that for this example $\partial_1 g^{\alpha\beta} = \eta^{\alpha\beta}$. This means that the metrics η and g form a flat pencil: the linear combination $g^{-1} - \lambda \eta^{-1}$ is a flat metric $\forall \lambda$. We will return to this observation presently. For now, let us note the following useful lemma for computing the canonical coordinates of a Frobenius manifold using the intersection form.

Lemma 3. [16] *On a semi-simple Frobenius manifold, the roots of the characteristic equation*

$$\det(g^{\alpha\beta}(t) - u\eta^{\alpha\beta}) = 0 \quad (1.4.18)$$

are simple, and are canonical coordinates for the Frobenius manifold. Conversely, if the roots of the characteristic equation are generically simple, then the Frobenius manifold is semisimple.

Proof. Recall the definition of the intersection form,

$$g(\omega_1, \omega_2) = \iota_E(\omega_1 \circ \omega_2),$$

and so in the canonical coordinate system we have

$$g^{ii} = u_i \eta^{ii}(u), \quad \text{for } \eta^{ii} = \eta_{ii}^{-1}.$$

Hence the equation (1.4.18) reads

$$\prod_{i=1}^N (u - u_i) = 0.$$

□

Example 11. For the Frobenius manifold defined by the data (1.2.14), the canonical coordinates are

$$u_1 = t_1 + 2\sqrt{t_2}, \quad u_2 = t_1 - 2\sqrt{t_2}.$$

In this coordinate system, the metric reads

$$\eta_{ij} = \begin{pmatrix} \frac{1}{8}(u_1 - u_2) & 0 \\ 0 & -\frac{1}{8}(u_1 - u_2) \end{pmatrix}.$$

Note that the Egoroff potential is $\Phi = \frac{1}{16}(u_1 - u_2)^2$.

Example 12 (Continued from Example 5, page 12). For the Frobenius defined by the data

$$E = t^1 \frac{\partial}{\partial t^1} + \frac{1}{2} t^2 \frac{\partial}{\partial t^2}, \quad F(t^1, t^2, t^3) = \frac{1}{2} (t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{(t^2)^4}{16} \gamma(t^3), \quad E = t^1 \frac{\partial}{\partial t^1} + \frac{1}{2} t^2 \frac{\partial}{\partial t^2},$$

recall that the WDVV equations are equivalent to the nonlinear differential equation

$$\gamma''' = 6\gamma\gamma'' - 9(\gamma')^2. \quad (1.4.19)$$

Under the identification

$$\gamma(\tau) = -\frac{2}{3}(\omega_1(\tau) + \omega_2(\tau) + \omega_3(\tau)),$$

the Chazy equation is equivalent to the following dynamical system ($\dot{\cdot} \equiv \frac{d}{dt_3}$)

$$\begin{aligned} \dot{\omega}_1 &= -\omega_2\omega_3 + \omega_1(\omega_2 + \omega_3), \\ \dot{\omega}_2 &= -\omega_3\omega_1 + \omega_2(\omega_3 + \omega_1), \\ \dot{\omega}_3 &= -\omega_1\omega_2 + \omega_3(\omega_1 + \omega_2), \end{aligned} \quad (1.4.20)$$

on the functions $\{\omega_1, \omega_2, \omega_3\}$. The system (2.3.3) was originally discovered by Halphen [33], and was re-discovered by Atiyah and Hitchin in the study of metrics on monopole moduli spaces [2]. We will call it the Halphen system. The canonical coordinates are given by

$$u_i(t_1, t_2, t_3) = t_1 + \frac{1}{2} t_2^2 \omega_i(t_3), \quad \text{for } i = 1, 2, 3. \quad (1.4.21)$$

In this coordinate system the metric reads

$$\eta_{ij} = \frac{1}{4} t_2^2 \begin{pmatrix} (u_1 - u_2)^{-1} (u_1 - u_3)^{-1} & 0 & 0 \\ 0 & (u_2 - u_1)^{-1} (u_2 - u_3)^{-1} & 0 \\ 0 & 0 & (u_1 - u_3)^{-1} (u_2 - u_3)^{-1} \end{pmatrix}.$$

Lemma 4. [16] The curvature of the intersection form is zero.

Proof. In canonical coordinates, the covariant components of the intersection form read

$$g_{ij} = u_i \eta_{ii} \delta_{ij}.$$

Denoting the rotation coefficients for g by

$$\tilde{\beta}_{ij} = \frac{\partial_j \sqrt{g_{ii}}}{\sqrt{g_{jj}}},$$

we have

$$\tilde{\beta}_{ij} = \frac{\partial_j \sqrt{u_i \eta_{ii}}}{\sqrt{u_j \eta_{jj}}} = \sqrt{\frac{u_i}{u_j}} \beta_{ij}, \quad (1.4.22)$$

where the β_{ij} are the rotation coefficients for the metric η . As for the metric η , because the intersection form diagonalises in the canonical coordinates, flatness of g is equivalent to the vanishing of

$$\begin{aligned} \tilde{R}_{il}^{ij} &= \frac{1}{\sqrt{g_{ii}}\sqrt{g_{jj}}} \left(\partial_l \tilde{\beta}_{ji} - \tilde{\beta}_{jl} \tilde{\beta}_{li} \right), \\ \tilde{R}_{ij}^{ij} &= \frac{1}{\sqrt{g_{ii}}\sqrt{g_{jj}}} \left(\partial_i \tilde{\beta}_{ij} + \partial_j \tilde{\beta}_{ji} + \sum_{p \neq i,j} \tilde{\beta}_{pj} \tilde{\beta}_{pi} \right). \end{aligned}$$

We have, on using (1.4.22), (recall that i, j , and l are distinct)

$$\tilde{R}_{il}^{ij} = \frac{1}{\sqrt{g_{ii}}\sqrt{g_{jj}}} \left(\partial_l \tilde{\beta}_{ji} - \tilde{\beta}_{jl} \tilde{\beta}_{li} \right) = \frac{1}{\sqrt{u_i u_j} \sqrt{\eta_{ii}} \sqrt{\eta_{jj}}} \left(\sqrt{\frac{u_i}{u_j}} \partial_l \beta_{ji} - \sqrt{\frac{u_l}{u_j}} \beta_{jl} \sqrt{\frac{u_i}{u_l}} \eta_{li} \right) = \frac{1}{u_j} R_{il}^{ij}, \quad (1.4.23)$$

and so is zero if the curvature of η is zero. Similary, we find

$$\begin{aligned} \tilde{R}_{ij}^{ij} &= \frac{1}{\sqrt{g_{ii}}\sqrt{g_{jj}}} \left(\partial_i \tilde{\beta}_{ij} + \partial_j \tilde{\beta}_{ji} + \sum_{p \neq i,j} \tilde{\beta}_{pj} \tilde{\beta}_{pi} \right) \\ &= \frac{1}{\sqrt{u_i u_j} \sqrt{\eta_{ii}} \sqrt{\eta_{jj}}} \left(\partial_i \left(\sqrt{\frac{u_i}{u_j}} \beta_{ij} \right) + \partial_j \left(\sqrt{\frac{u_j}{u_i}} \beta_{ji} \right) + \sum_{p \neq i,j} \left(\sqrt{\frac{u_p}{u_j}} \beta_{pj} \right) \left(\sqrt{\frac{u_i}{u_p}} \beta_{pi} \right) \right) \\ &= \frac{1}{\sqrt{u_i u_j} \sqrt{\eta_{ii}} \sqrt{\eta_{jj}}} \left(\sqrt{\frac{u_i}{u_j}} \partial_i \beta_{ij} + \frac{1}{2\sqrt{u_i u_j}} \beta_{ij} + \sqrt{\frac{u_j}{u_i}} \partial_j \beta_{ji} + \frac{1}{2\sqrt{u_i u_j}} \beta_{ji} + \sum_{p \neq i,j} \frac{u_p}{\sqrt{u_i u_j}} \beta_{pj} \beta_{pi} \right) \\ &= \frac{1}{u_i u_j \sqrt{\eta_{ii}} \sqrt{\eta_{jj}}} \left(u_i \partial_i \beta_{ij} + u_j \partial_j \beta_{ji} + \frac{1}{2} \beta_{ij} + \frac{1}{2} \beta_{ji} + \sum_{p \neq i,j} u_p \beta_{pj} \beta_{pi} \right). \end{aligned}$$

Using the vanishing of the curvature of R_{il}^{ij} , we have $\beta_{pj} \beta_{pi} = \partial_p \beta_{ji}$. Using this, together with the Egoroff property of the metric η , $\beta_{ij} = \beta_{ji}$, we have

$$\tilde{R}_{ij}^{ij} = \frac{1}{u_i u_j \sqrt{\eta_{ii}} \sqrt{\eta_{jj}}} \left(\sum_{p=1}^N u_p \partial_p \beta_{ij} + \beta_{ij} \right). \quad (1.4.24)$$

Recall also that the symmetry of the rotation coefficients, combined with the vanishing of the components R_{ij}^{ij} of the Riemann tensor gave

$$\sum_{p=1}^N u_p \partial_p \beta_{ij} = -\beta_{ij}.$$

The lemma is proved. □

1.4.2 Flat Pencils of Metrics and Frobenius Manifolds

We have seen that on a Frobenius manifold we can define two flat metrics η and g . We saw in the proof of Lemma 4 that the vanishing of the curvature of g depended not just on the flatness of η , but also on the the covariant constancy of the unity field, and covariant linearity of the Euler field (recall that these led to a refinement of the Darboux equations and homogeneity properties of the rotation coefficients). It would appear then, by the converse of Proposition 1, that having a pair of flat metrics on a manifold \mathcal{M} , whose curvature is dependent upon one another (for example, via equations (1.4.23), (1.4.24)) is rather close to the existence of a Frobenius manifold.

One way to make this connection more explicit would be to carry out the curvature calculations for the intersection form in the flat coordinates $\{t^\alpha : \alpha = 1, \dots, N\}$ for the metric η . In this coordinate system, the contravariant Christoffel symbols (defined by $\Gamma_\gamma^{\alpha\beta} = -g^{\alpha\sigma}\Gamma_{\sigma\gamma}^\beta$) of the the intersection form read [18]

$${}^g\Gamma_\gamma^{\alpha\beta}(t) = \left(\frac{1-d}{2} \delta_\gamma^\varepsilon - (\nabla E)_\gamma^\varepsilon \right) c_\varepsilon^{\alpha\beta}(t). \quad (1.4.25)$$

Then the calculation for $\tilde{R} = 0$ uses associativity, potentiality, and homogeneity. The notion that formalises this interplay between the two metrics is that of a *flat pencil* of metrics.

The following is an amalgamation of results, slightly recast or trimmed to suit our needs. For a more comprehensive introduction, we refer the reader to [18].

Definition 5. Two contravariant metrics $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ form a flat pencil if:

1) The linear combination

$$(\cdot, \cdot)_1 - \lambda(\cdot, \cdot)_2 \quad (1.4.26)$$

is a contravariant metric for all values of λ .

2) If ${}^1\Gamma_\gamma^{\alpha\beta}$ and ${}^2\Gamma_\gamma^{\alpha\beta}$ are the contravariant components of their corresponding Levi-Civita connections, then the linear combination

$${}^1\Gamma_\gamma^{\alpha\beta} - \lambda \cdot {}^2\Gamma_\gamma^{\alpha\beta}$$

gives the components of the Levi-Civita connection for the metric (1.4.26).

3) The metric (1.4.26) is flat for all λ .

Lemma 5. If for a flat metric in some coordinate system $\{x^i : i = 1, \dots, N\}$, both the components of the metric $g^{\alpha\beta}(x)$, and the components of the connection $\Gamma_\gamma^{\alpha\beta}$ depend at

most linearly on the coordinate x^1 , then the metrics

$$g_1^{\alpha\beta} := g^{\alpha\beta} \quad \text{and} \quad g_2^{\alpha\beta} = \frac{\partial}{\partial x^1} g^{\alpha\beta}$$

form a flat pencil assuming that $\det(g_2^{\alpha\beta}) \neq 0$. The correspondent Levi-Civita connections have the form

$${}^1\Gamma_\gamma^{\alpha\beta} = \Gamma_\gamma^{\alpha\beta}, \quad {}^2\Gamma_\gamma^{\alpha\beta} = \frac{\partial}{\partial x^1} \Gamma_\gamma^{\alpha\beta}.$$

Proof. Is straightforward and will be omitted, the reader is referred to [16], Appendix D. \square

Proposition 2. *On a Frobenius manifold, the metrics η and g form a flat pencil.*

Proof. This will be done by showing that the bilinear form

$$P = g^{-1} - \lambda \eta^{-1}$$

is in fact a metric, and satisfies the criteria of Lemma 5. To see that P is non-degenerate, note that for Euler vector fields of the form (1.2.10) (i.e. $d_1 = 1$), we have chosen the flat coordinates in such a way that

$$g^{\alpha\beta}(t) = E^1 c_1^{\alpha\beta} + \sum_{\sigma=2}^N E^\sigma c_\sigma^{\alpha\beta} = t^1 \eta^{\alpha\beta} + \tilde{g}^{\alpha\beta}(t^2, \dots, t^N). \quad (1.4.27)$$

Hence,

$$g^{\alpha\beta}(t) - \lambda \eta^{\alpha\beta} = \eta^{\alpha\beta}(t^1 - \lambda) + \tilde{g}^{\alpha\beta}(t^2, \dots, t^N) \quad (1.4.28)$$

is non-degenerate for all λ . Equation (1.4.27) also shows that g depends linearly on t^1 . Note also that

$$\frac{\partial}{\partial t^1} g \Gamma_\gamma^{\alpha\beta} = \frac{\partial}{\partial t^1} \left\{ \left(\frac{d-1}{2} \delta_\gamma^\varepsilon + (\nabla E)_\gamma^\varepsilon \right) c_\gamma^{\alpha\beta} \right\} = 0,$$

since $c_{\gamma^1}^{\alpha\beta} = 0$. It remains to show that the curvature of the metric P is zero. Recall that the contravariant components of the Levi-Civita connection of a metric g are the solutions to the equations

$$\partial_\gamma g^{\alpha\beta} = \Gamma_\gamma^{\alpha\beta} + \Gamma_\gamma^{\beta\alpha} \quad (1.4.29)$$

$$g^{\alpha\varepsilon} \Gamma_\varepsilon^{\beta\kappa} = g^{\beta\varepsilon} \Gamma_\varepsilon^{\alpha\kappa}. \quad (1.4.30)$$

In the flat coordinates of the metric η , we have

$$\frac{\partial}{\partial t^\gamma} P^{\alpha\beta} = \frac{\partial}{\partial t^\gamma} (g^{\alpha\beta} - \lambda \eta^{\alpha\beta}) = \frac{\partial}{\partial t^\gamma} g^{\alpha\beta} = g \Gamma_\gamma^{\alpha\beta} + g \Gamma_\gamma^{\beta\alpha}.$$

The latter equation (1.4.30) for the pencil of metrics reads

$$(g^{\alpha\varepsilon}(t) - \lambda \eta^{\alpha\varepsilon})^P \Gamma_\varepsilon^{\beta\kappa} = (g^{\beta\varepsilon}(t) - \lambda \eta^{\beta\varepsilon})^P \Gamma_\varepsilon^{\alpha\kappa}.$$

Or, recalling equation (1.4.28),

$$(\eta^{\alpha\varepsilon}(t^1 - \lambda) + \tilde{g}^{\alpha\varepsilon}(t^2, \dots, t^N))^P \Gamma_\varepsilon^{\beta\kappa} = (\eta^{\beta\varepsilon}(t^1 - \lambda) + \tilde{g}^{\beta\varepsilon}(t^2, \dots, t^N))^P \Gamma_\varepsilon^{\alpha\kappa}.$$

Equating coefficients of t^1 implies

$$(\eta^{\alpha\varepsilon})^P \Gamma_\varepsilon^{\beta\kappa} = (\eta^{\beta\varepsilon})^P \Gamma_\varepsilon^{\alpha\kappa} \quad \Rightarrow \quad (g^{\alpha\varepsilon})^P \Gamma_\varepsilon^{\beta\kappa} = (g^{\beta\varepsilon})^P \Gamma_\varepsilon^{\alpha\kappa}.$$

Hence

$${}^P \Gamma_\gamma^{\alpha\beta}(t) = {}^S \Gamma_\gamma^{\alpha\beta}(t) \quad \Rightarrow \quad {}^P R_{\beta\gamma\delta}^\alpha = {}^S R_{\beta\gamma\delta}^\alpha.$$

But ${}^S R_{\beta\gamma\delta}^\alpha = 0$. The lemma is proved. \square

Therefore, given a Frobenius manifold, one can always construct a flat pencil of metrics. It turns out that one can (almost) go in the other direction as well - we did not use the homogeneity properties of the metric in deriving the zero curvature of P . One needs some extra assumptions, namely that the metrics form a *quasihomogeneous* flat pencil.

Definition 6. A flat pencil of metrics $(\cdot, \cdot)_1 - (\cdot, \cdot)_2$ is said to be *quasihomogeneous* if there exists a function τ on \mathcal{M} such that the vector fields $E := {}^1 \nabla \tau$, $e := {}^2 \nabla \tau$ satisfy

- 1) $[e, E] = e$,
- 2) $\mathcal{L}_E(\cdot, \cdot)_1 = (d-1)(\cdot, \cdot)_1$,
- 3) $\mathcal{L}_e(\cdot, \cdot)_1 = (\cdot, \cdot)_2$,
- 4) $\mathcal{L}_e(\cdot, \cdot)_2 = 0$.

Choosing $(\cdot, \cdot)_1 = g$ and $(\cdot, \cdot)_2 = \eta$, we obtain from an arbitrary Frobenius manifold a quasihomogeneous flat pencil. Thus, we have one direction of

Theorem 3. Every Frobenius manifold carries a natural quasihomogeneous flat pencil; conversely every regular quasihomogeneous flat pencil on a manifold endows it with the structure of a Frobenius manifold.

To construct a Frobenius manifold from a flat pencil of metrics, one first observes that the differences

$$\Delta^{\alpha\beta\gamma} := {}^1 \Gamma_\varepsilon^{\beta\gamma} g_1^{\alpha\varepsilon} - {}^2 \Gamma_\sigma^{\alpha\gamma} g_2^{\beta\sigma}$$

are the components of a tensor. From this one constructs a bilinear operation

$$\Delta : \Gamma(T^*\mathcal{M}, \mathcal{M}) \times \Gamma(T^*\mathcal{M}, \mathcal{M}) \rightarrow \Gamma(T^*\mathcal{M}, \mathcal{M}), \quad (\zeta, \xi) \mapsto \Delta(\zeta, \xi), \quad \text{for } \zeta, \xi \in \Gamma(T^*\mathcal{M}, \mathcal{M}).$$

In local coordinates $\{x^\alpha : \alpha = 1, \dots, N\}$,

$$\Delta(\zeta, \xi)_\sigma = \sum_{\alpha, \beta=1}^N \zeta_\alpha \xi_\beta \Delta^{\alpha\beta\gamma} g_{2\gamma\sigma}.$$

The adjective regular means the endomorphism of the tangent bundle whose components in the flat coordinates of the metric $(\cdot, \cdot)_2$ are given by

$$R_\beta^\alpha = \frac{d-1}{2} \delta_\beta^\alpha + {}^2\nabla_\beta E^\alpha \quad (1.4.31)$$

is non-degenerate. One obtains a unique Frobenius structure on \mathcal{M} by defining

$$\zeta \circ \xi = \Delta(\zeta, R^{-1}\xi).$$

As we will see in forthcoming chapters, owing to the work of Dubrovin and Novikov on the bi-Hamiltonian structure of partial differential equations, the theory of flat pencils of metrics gives a direct link between integrable PDEs of hydrodynamic type and Frobenius manifolds. Flat pencils of metrics can also be used to endow the orbit space of a finite Coxeter group with the structure of a Frobenius manifold.

1.5 The Inversion Symmetry

In this section we outline a symmetry of the WDVV equations that lies at the heart of this thesis. Verlinde and Warner's 1991 paper [66] studied the families of topological field theories described by the Landau-Ginzburg superpotential

$$\begin{aligned} \lambda(z_1, z_2, z_3; \mathbf{t}) = & -\frac{1}{3} (z_1^3 + z_2^3 + z_3^3) + \alpha_1(t_8) z_1 z_2 z_3 + \alpha_2(t_8) (t_5 z_1 z_2 + t_6 z_1 z_3 + t_7 z_2 z_3) \\ & + \alpha_3(t_8) (t_2 z_1 + t_3 z_2 + t_4 z_3) + \alpha_4(t_8) (t_5 t_6 z_1 + t_5 t_7 z_2 + t_6 t_7 z_3) \\ & + \frac{1}{2} \alpha_5(t_8) (t_7^2 z_1 + t_6^2 z_2 + t_5^2 z_3) + \alpha_6(t_8) (t_2 t_7 + t_3 t_6 + t_4 t_5) \\ & + \frac{1}{6} \alpha_7(t_8) (t_5^3 + t_6^3 + t_7^3) + \alpha_8(t_8) t_5 t_6 t_7 + t_1. \end{aligned}$$

The parameters $\{t_\alpha : \alpha = 1, \dots, 8\}$ are interpreted as moduli describing deformations of the theory defined by the potential $\lambda(z_1, z_2, z_3; 0)$. They found the Frobenius manifold

structure on the so-called Chiral ring of primary fields of the theory,

$$A = \frac{\mathbb{C}[z_1, z_2, z_3]}{\langle d\lambda = 0 \rangle}.$$

Note that the algebra at the origin (the point where all the moduli are equal to zero) is the Jacobi ring of the singularity of $\lambda(z_1, z_2, z_3; 0)$ at $z_1 = z_2 = z_3 = 0$. This is an example of a phenomenon called *mirror symmetry*. Verlinde and Warner found the following solution to WDVV

$$\begin{aligned} F = & \frac{1}{2}t_1^2t_8 + t_1(t_2t_7 + t_3t_6 + t_4t_5) + t_2t_3t_4f_0(t_8) + \frac{1}{6}(t_2^3 + t_3^3 + t_4^3)f_1(t_8) \\ & + (t_2t_3t_6t_7 + t_2t_4t_5t_7 + t_3t_4t_5t_6)f_2(t_8) + \frac{1}{2}(t_2^2t_5t_6 + t_3^2t_5t_7 + t_4^2t_6t_7)f_3(t_8) \\ & + (t_2t_3t_5^2 + t_2t_4t_6^2 + t_3t_4t_7^2)f_4(t_8) + \frac{1}{4}(t_2^2t_7^2 + t_3^2t_6^2 + t_4^2t_5^2)f_5(t_8) \\ & + \frac{1}{6}[t_2t_7(t_5^3 + t_6^3) + t_3t_6(t_5^3 + t_7^3) + t_4t_5(t_6^3 + t_7^3)]f_6(t_8) \\ & + \frac{1}{2}(t_2t_5t_6t_7^2 + t_3t_5t_6^2t_7 + t_3t_5^2t_6t_7)f_7(t_8) + \frac{1}{4}(t_2t_5^2t_6^2 + t_3t_5^2t_7^2 + t_4t_6^2t_7^2)f_8(t_8) \\ & + \frac{1}{24}(t_2t_7^4 + t_3t_6^4 + t_4t_5^4)f_9(t_8) + \frac{1}{36}(t_5^3t_6^3 + t_5^3t_7^3 + t_6^3t_7^3)f_{10}(t_8) \\ & + \frac{1}{24}(t_5t_6t_7^4 + t_5t_6^4t_7 + t_5^4t_6t_7)f_{11}(t_8) + \frac{1}{8}t_5^2t_6^2t_7^2f_{12}(t_8) + \frac{1}{720}(t_5^6 + t_6^6 + t_7^6)f_{13}(t_8), \end{aligned} \quad (1.5.1)$$

with Euler field

$$E = t^1 \frac{\partial}{\partial t^1} + \frac{2}{3} \left(t^2 \frac{\partial}{\partial t^2} + t^3 \frac{\partial}{\partial t^3} + t^4 \frac{\partial}{\partial t^4} \right) + \frac{1}{3} \left(t^5 \frac{\partial}{\partial t^5} + t^6 \frac{\partial}{\partial t^6} + t^7 \frac{\partial}{\partial t^7} \right).$$

The functions f_i may all be expressed in terms of the Schwarzian triangle function $S[\frac{1}{2}, \frac{1}{2}, \frac{1}{6}, t]$ (for the definition of the Schwarzian triangle function, see Chapter 2). Interestingly, this solution was found before the formal definition of a Frobenius manifold.

The functions $\{f_0, \dots, f_{13}\}$ appearing in their solution were shown to have the following transformation properties under $t_8 \rightarrow -1/t_8$:

$$f_i \left(-\frac{1}{t_8} \right) = \begin{cases} t_8^2 f_i(t_8) + 2t_8, & \text{for } i = 2, 5, \\ t_8 f_i(t_8), & \text{for } i = 0, 1, \\ t_8^2 f_i(t_8), & \text{for } i = 3, 4, \\ t_8^3 f_i(t_8), & \text{for } i = 6, 7, 8, 9, \\ t_8^4 f_i(t_8), & \text{for } i = 10, 11, 12, 13. \end{cases} \quad (1.5.2)$$

This remarkably rich structure led Verlinde and Warner to make the following remarks [66]:

“There are one or two surprises in the foregoing results. Most particularly λ and F are not manifestly modular invariant... To render λ completely modular invariant one must require that under $t_8 \mapsto (at_8 + b)/(ct_8 + d)$ the coupling constant t_1 transforms according to

$$t_1 \mapsto t_1 + \frac{c(ct_8 + d)}{ad - bc}(t_2t_7 + t_3t_6 + t_4t_5). \quad (1.5.3)$$

This then renders λ completely modular invariant. The modular properties of F are more vexatious. Based on the transformation properties of the cosmological constant one finds that F should have weight -1 , and indeed this is consistent with the weights of the f_n . There is also a modular anomaly in f_2 and f_5 , akin to the one in α_6 . This anomaly can almost be cancelled by a transformation of the form (1.5.3). The real problem is, however, with the very first term in F , which manifestly cannot be rendered modular invariant without modifying F . This raises the question as to whether F should be modular invariant since it is a prepotential... The fact that λ should be modular invariant is intuitively clear, but the modular anomaly and its cancellation by (1.5.3) is somewhat unexpected.”

The answer to their question as to how the prepotential should transform is given by

$$\begin{aligned} t_1 \mapsto \hat{t}_1 &:= t_1 + \frac{1}{2t_8}(t_2t_7 + t_3t_6 + t_4t_5), \\ t_i \mapsto \hat{t}_i &:= \frac{t_i}{t_8}, \quad \text{for } i \neq 1, N, \\ t_8 \mapsto \hat{t}_8 &:= -\frac{1}{t_8}, \\ F(t) \mapsto \hat{F}(\hat{\mathbf{t}}) &:= F(\mathbf{t}(\hat{\mathbf{t}})) - \frac{1}{2}t_1(t_1t_8 + t_2t_7 + t_3t_6 + t_4t_5) \end{aligned}$$

Then the transformation properties (1.5.2) mean that under this transformation we have

$$\hat{F}(\hat{\mathbf{t}}) = F(\hat{\mathbf{t}})$$

The structure that Verlinde and Warner had found is what we will come to know as a *modular Frobenius manifold*, and will be the subject of the next chapter. Let us elevate the symmetry observed here to a definition:

Definition 7. [16] *The inversion symmetry is defined by*

$$\hat{t}^1 = \frac{1}{2} \frac{t_\sigma t^\sigma}{t^N}, \quad \hat{t}^\alpha = \frac{t^\alpha}{t^N}, \quad (\text{for } \alpha \neq 1, N), \quad \hat{t}^N = -\frac{1}{t^N},$$

$$\hat{F}(\hat{t}) = (\hat{t}^N)^2 F\left(\frac{1}{2} \frac{\hat{t}_\sigma \hat{t}^\sigma}{\hat{t}^N}, -\frac{\hat{t}^2}{\hat{t}^N}, \dots, -\frac{\hat{t}^{N-1}}{\hat{t}^N}, -\frac{1}{\hat{t}^N}\right) + \frac{1}{2} \hat{t}^1 \hat{t}_\sigma \hat{t}^\sigma, \quad (1.5.4)$$

$$\hat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}.$$

It can be shown by direct calculation [16] that the structure constants of the inverted Frobenius manifold are related to those of the original by

$$\hat{c}_{\alpha\beta\gamma}(\hat{\mathbf{t}}) = (t^N)^{-2} \frac{\partial t^\lambda}{\partial \hat{t}^\alpha} \frac{\partial t^\mu}{\partial \hat{t}^\beta} \frac{\partial t^\nu}{\partial \hat{t}^\gamma} c_{\lambda\mu\nu}(\mathbf{t}(\hat{\mathbf{t}})), \quad (1.5.5)$$

which shows that the inversion symmetry is a genuine symmetry of WDVV: it maps solutions to solutions.

Under the assumption that $r_i = 0$ for $i = 1, \dots, N$ the corresponding Euler vector field of the inverted Frobenius manifold \hat{F} has the form

$$\hat{E}(\hat{t}) = \sum_{\alpha} (1 - \frac{\hat{d}}{2} - \hat{\mu}_{\alpha}) \hat{t}^{\alpha} \hat{\partial}_{\alpha}, \quad (1.5.6)$$

where

$$\hat{d} = 2 - d, \quad \hat{\mu}_1 = \mu_N - 1, \quad \hat{\mu}_N = \mu_1 + 1, \quad \hat{\mu}_i = \mu_i, \quad \text{for } i \neq 1, N. \quad (1.5.7)$$

Example 13. [Continued from Example 5, page 12] Suppose $N = 3$ and let the Euler vector field and prepotential take the form

$$E = t^1 \frac{\partial}{\partial t^1} + \frac{1}{2} t^2 \frac{\partial}{\partial t^2}, \quad F(t^1, t^2, t^3) = \frac{1}{2} (t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{(t^2)^4}{16} \gamma(t^3). \quad (1.5.8)$$

The WDVV equations are equivalent to the nonlinear differential equation (known as the Chazy equation)

$$\gamma''' = 6\gamma\gamma'' - 9(\gamma')^2 \quad (1.5.9)$$

for the function $\gamma(t^3)$ (here $'$ denotes differentiation with respect to the variable t^3). The equation has an $SL(2, \mathbb{C})$ invariance, mapping solutions to solutions,

$$t^3 \mapsto \hat{t}^3 = \frac{at^3 + b}{ct^3 + d}, \quad \gamma(t^3) \mapsto \hat{\gamma}(\hat{t}^3) = (ct^3 + d)^2 \gamma(t^3) + 2c(ct^3 + d), \quad ad - bc = 1. \quad (1.5.10)$$

A simple calculation shows that under the inversion symmetry

$$\begin{aligned} \hat{F}(\hat{t}) &= \frac{1}{2} (\hat{t}^1)^2 \hat{t}^3 + \frac{1}{2} \hat{t}^1 (\hat{t}^2)^2 - \frac{(\hat{t}^2)^4}{16} \left\{ \frac{1}{(\hat{t}^3)^2} \gamma\left(\frac{-1}{\hat{t}^3}\right) - \frac{2}{\hat{t}^3} \right\}, \\ &= \frac{1}{2} (\hat{t}^1)^2 \hat{t}^3 + \frac{1}{2} \hat{t}^1 (\hat{t}^2)^2 - \frac{(\hat{t}^2)^4}{16} \hat{\gamma}(\hat{t}^3) \end{aligned}$$

where, on using the symmetry given by (1.5.10), $\hat{\gamma}$ is also a solution of the Chazy equation. Thus under inversion symmetry one has a weak modular symmetry: the functional form

of the prepotential is preserved, but with the function γ being replaced by a new solution of the same equation connected by a special case of the symmetry (1.5.10).

The inversion symmetry also maps distinct solutions of WDVV to one another:

Example 14. Recall that the free energy and Euler vector field

$$F = \frac{1}{2}t_1^2t_2 + t_2^2 \log t_2; \quad E = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2}$$

defines a two-dimensional Frobenius manifold. Applying the inversion symmetry, we find

$$\begin{aligned} \hat{t}_2^2 F\left(\hat{t}_1, -\frac{1}{\hat{t}_2}\right) + \frac{1}{2}\hat{t}_1(\hat{t}_1\hat{t}_2 + \hat{t}_2\hat{t}_1) &= \hat{t}_2^2 \left(\frac{1}{2}\hat{t}_1^2 \left(-\frac{1}{\hat{t}_2}\right) + \left(-\frac{1}{\hat{t}_2}\right)^2 \log\left(-\frac{1}{\hat{t}_2}\right) \right) \\ &= \frac{1}{2}\hat{t}_1^2\hat{t}_2 - \log(\hat{t}_2), \end{aligned}$$

since F is only defined up to the addition of a quadratic. The Euler field transforms as

$$E = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} = \hat{t}_1 \frac{\partial}{\partial \hat{t}_1} - \frac{2}{\hat{t}_2} \left(\hat{t}_2^2 \frac{\partial}{\partial \hat{t}_2} \right) = \hat{t}_1 \frac{\partial}{\partial \hat{t}_1} - 2\hat{t}_2 \frac{\partial}{\partial \hat{t}_2}.$$

Recalling the list of two dimensional Frobenius manifolds, we see that the solution (1.2.14) is mapped to (1.2.15),

$$\hat{F} = \frac{1}{2}\hat{t}_1^2\hat{t}_2 - \log \hat{t}_2; \quad \hat{E} = \hat{t}_1 \frac{\partial}{\partial \hat{t}_1} - 2\hat{t}_2 \frac{\partial}{\partial \hat{t}_2}.$$

As mentioned earlier, we will come to see that these two Frobenius manifolds describe the Bi-Hamiltonian structures of the dispersionless limits of the Benney and Harry Dym hierarchies respectively. Investigating how various structures on the respective Frobenius manifolds are related will lead to an understanding of how the inversion symmetry lifts to these two hierarchies of partial differential equations of hydrodynamic type.

1.6 Examples of Frobenius Manifolds

In this section we will describe a couple of well known examples of Frobenius manifolds.

1.6.1 Rational Maps of Degree $N + 1$

This example follows on from Example 2, and is originally due to Dubrovin. The Frobenius manifold structure is given on the space of versal deformations of the singularity.

Let \mathcal{M} be the space of complex polynomials of the form

$$p(z) = z^{N+1} + a_N z^N + \dots + a_1, \quad a_i \in \mathbb{C}$$

The tangent space is identified with the space of polynomials of degree strictly less than N :

$$T_p \mathcal{M} \cong \frac{\mathbb{C}[z]}{\langle p'(z) \rangle}.$$

This algebra has dimension N as a vector space: it is spanned by the monomials $\{1, z, \dots, z^{N-1}\}$. So tangent vectors at p have the form

$$\dot{p}(z) = \dot{a}_N z^N + \dots + \dot{a}_1.$$

We define the inner product on $T_p \mathcal{M}$ by

$$\eta(\dot{p}, \dot{q}) = -\text{Res}_{z=\infty} \frac{\dot{p}(z)\dot{q}(z)}{p'(z)} dz.$$

Then \mathcal{M} is a Frobenius manifold with canonical coordinates given by the critical points of p :

$$u_i = p(\alpha_i), \quad \text{for } p'(\alpha_i) = 0.$$

Hence this Frobenius manifold will be semi-simple if and only if the polynomial $p'(z)$ has no multiple roots, which is generically true. Those submanifolds where this assumption fails are known as caustics. Note that the tangent space at the origin of the coordinate system $\{a_i : i = 1, \dots, N\}$ coincides with the algebra presented in Example 2.

The Euler vector field and unity take the form

$$E = \frac{1}{N+1} \sum_k (N-k+1) a_k \frac{\partial}{\partial a_k}, \quad e = \frac{\partial}{\partial a_1}.$$

To prove this assertion, we will show that the metric is flat and Egoroff, with potential

$$\Phi = \frac{a_N}{N+1}$$

and apply the converse of Proposition 1. First, note that

$$\begin{aligned}\delta_{ij} = \frac{\partial u_i}{\partial u_j} &= \frac{\partial}{\partial u_j} p(\alpha_i) = \frac{\partial p}{\partial u_j}(\alpha_i) + p'(\alpha_i) \frac{\partial \alpha_i}{\partial u_j} \\ &= \frac{\partial p}{\partial u_j}(\alpha_i), \quad \text{as } p'(\alpha_i) = 0.\end{aligned}$$

Therefore by Lagrange interpolation we have

$$\frac{\partial p}{\partial u_j} = \prod_{i \neq j} \frac{z - \alpha_i}{\alpha_j - \alpha_i}.$$

Therefore $p'(z) = (N+1)(z - \alpha_1) \dots (z - \alpha_N)$ divides the product $\frac{\partial p}{\partial u_i} \frac{\partial p}{\partial u_j}$, and so the residue vanishes for $i \neq j$:

$$\eta \left(\frac{\partial p}{\partial u_i}, \frac{\partial p}{\partial u_j} \right) = -\text{Res}_{z=\infty} \frac{\partial p}{\partial u_i} \frac{\partial p}{\partial u_j} \frac{dz}{p'} = 0.$$

If, on the other hand $i = j$,

$$\begin{aligned}\eta_{ii} &= -\text{Res}_{z=\infty} \frac{\left(\prod_{j \neq i} \frac{z - \alpha_j}{\alpha_i - \alpha_j} \right)^2}{(N+1) \prod_k (z - \alpha_k)} dz \\ &= -\text{Res}_{z=\infty} \frac{\prod_{j \neq i} \frac{z - \alpha_j}{(\alpha_i - \alpha_j)^2}}{(N+1)(z - \alpha_i)} dz.\end{aligned}$$

Note that this differential has two poles, at $z = \infty$ and $z = \alpha_i$, and on a compact Riemann surface (such as \mathbb{P}^1), the sum of the residues of a meromorphic differential is equal to zero. Hence

$$\begin{aligned}-\text{Res}_{z=\infty} \frac{\prod_{j \neq i} \frac{z - \alpha_j}{(\alpha_i - \alpha_j)^2}}{(N+1)(z - \alpha_i)} dz &= \text{Res}_{z=\alpha_i} \frac{\prod_{j \neq i} \frac{z - \alpha_j}{(\alpha_i - \alpha_j)^2}}{(N+1)(z - \alpha_i)} \\ &= \frac{1}{N+1} \prod_{j \neq i} \frac{z - \alpha_j}{\alpha_i - \alpha_j} \Big|_{z=\alpha_i} \\ &= \frac{1}{N+1} \prod_{j \neq i} \frac{1}{\alpha_j - \alpha_i}.\end{aligned}$$

In order to see that the metric is Egoroff, equate factorised and unfactorised expressions for $\frac{\partial p}{\partial u_j}$:

$$\frac{\partial p}{\partial u_j} = \prod_{i \neq j} \frac{z - \alpha_i}{\alpha_j - \alpha_i} = \sum_{i=1}^N \frac{\partial a_i}{\partial u_j} z^{i-1},$$

and compare coefficients of z^{N-1} ,

$$\prod_{i \neq j} \frac{1}{\alpha_j - \alpha_i} = \frac{\partial a_N}{\partial u_j} \Rightarrow \eta_{ii} = \frac{1}{N+1} \frac{\partial a_N}{\partial u_i}.$$

That the identity element is given by $\partial/\partial a_1$ is obvious: multiplying any polynomial by a constant will give an isomorphic polynomial in the algebra,

$$\frac{\partial p}{\partial a_1} \frac{\partial p}{\partial a_i} = \frac{\partial p}{\partial a_i}, \quad \text{for } i = 1, \dots, N.$$

To see that the metric is flat, we can construct a flat coordinate system for η explicitly. This is done by introducing a new function

$$w := p(z)^{\frac{1}{N+1}}.$$

Near $z = \infty$, we obtain the Puiseux series

$$z(w, t) = w + \frac{t_1}{w} + \frac{t_2}{w^2} + \dots + \frac{t_N}{w^N} + \mathcal{O}\left(\frac{1}{w^{N+1}}\right)$$

for z . Each t_i is a polynomial in the coefficients $\{a_1, \dots, a_N\}$. Differentiating $p(z(w, t)) = w^{N+1}$, we obtain an expression for the vector fields ∂_{t_i} :

$$\frac{\partial p}{\partial t_i}(z(w, t)) + p'(z(w, t)) \frac{\partial z}{\partial t_i} = \frac{\partial p}{\partial t_i}(z(w, t)) + p'(z(w, t)) \frac{1}{w^i} = 0.$$

Hence

$$\begin{aligned} \eta\left(\frac{\partial p}{\partial t_i}, \frac{\partial p}{\partial t_j}\right) &= -\text{Res}_{z=\infty} \left(\frac{\partial p}{\partial t_i} \frac{\partial p}{\partial t_j} \right) \frac{dz}{p'} = -\text{Res}_{z=\infty} \frac{p'(z)^2 dz}{w^{i+j} p'(z)} \\ &= -\text{Res}_{w=\infty} \frac{d(w^{N+1})}{w^{i+j}} = -\text{Res}_{w=\infty} (N+1) w^{N-i-j} dw = (N+1) \text{Res}_{v=0} v^{i+j-(N+2)} dv, \end{aligned}$$

for $w = 1/v$. This gives constant coefficients. It is also obvious that the metric is non-degenerate: the gram matrix in this coordinate system takes the form

$$\eta_{ij} = (N+1) \delta_{i+j, N+1}.$$

More explicitly, for the particular case

$$p(z) = z^4 + a_1 z^2 + a_2 z + a_1,$$

the tangent algebra is spanned by the monomials $f_1 = z^2$, $f_2 = z$, $f_3 = 1$. The inner product is given by the formula

$$\eta(f_i, f_j) = -4 \text{Res}_{z=\infty} \frac{f_i(z) f_j(z)}{4z^3 + 2a_1 z^2 + a_2} dz,$$

whose Gram matrix in the coordinate system $\{a_1, a_2, a_3\}$ is given by

$$\eta_{ij}(a) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -\frac{1}{2}a_1 \end{pmatrix}.$$

Introducing the coordinate system

$$\begin{aligned} a_3 &= t_1 + \frac{1}{8}t_3^2, \\ a_2 &= t_2, \\ a_1 &= t_3, \end{aligned}$$

we obtain constant coefficients,

$$\eta_{\alpha\beta}(t) = \sum_{s,r} \frac{\partial a_r}{\partial t_\alpha} \frac{\partial a_s}{\partial t_\beta} a_{sr}(a) = \delta_{i+j, N+1}.$$

The structure constants of the algebras are given by

$$c_{\alpha\beta\gamma}(t) = -\text{Res}_{z=\infty} \left(\frac{\partial p}{\partial t_\alpha} \frac{\partial p}{\partial t_\beta} \frac{\partial p}{\partial t_\gamma} \frac{dz}{p'(z)} \right).$$

One finds

$$c_{122} = 1, \quad c_{113} = 1, \quad c_{223} = -\frac{1}{4}t_3, \quad c_{233} = -\frac{1}{4}t_2, \quad c_{333} = \frac{1}{16}t_3^2,$$

which may be integrated to obtain the polynomial solution to WDVV

$$F = \frac{1}{2}t_1^2t_3 + \frac{1}{2}t_1t_2^2 - \frac{1}{16}t_2^2t_3^2 + \frac{1}{960}t_3^5. \quad (1.6.1)$$

This solution to WDVV can also be found as the unique Frobenius manifold structure on the space of orbits of the Coxeter group for the root system of type A_3 . The construction of the Frobenius structure on Coxeter group orbit spaces is our next example.

1.6.2 Coxeter Group Orbit Spaces

In this section we will outline the construction of an important class of Frobenius manifolds. They arise in the study of the differential geometry of the orbit spaces of finite Coxeter groups. It follows from the construction given in [16], Lecture 4 that the corresponding solutions to the WDVV equations will be polynomial in the flat coordinates $\{t^\alpha : \alpha = 1, \dots, N\}$. The solutions (1.2.13), (1.6.1), are examples.

Definition 8. A finite Coxeter group is a finite group of linear transformations of a real vector space V of dimension N , which is generated by reflections.

Less formally, dismantling a kaleidoscope is a good way to gain an understanding of what a Coxeter group is (at least for $N = 2$). Typically, a kaleidoscope will consist of a pair of mirrors intersecting at some special angle at its centre in such a way that any object between these two mirrors, once reflected, generates a finite number of images. That one sees a ‘nice’ image is due to the fact that the mirrors that one is looking down between intersect at an angle of the form $2\pi/k$, for $k \in \mathbb{Z}$. Most kaleidoscopes have $k = 6$, which generates an image with hexagonal symmetry. When $N > 2$, the alignments of mirrors that give nice images (i.e. that generate closed orbits) become a lot more special. Indeed in dimension 3, there are just three mirror arrangements. These correspond to the symmetries of the Platonic solids.

Let W be a finite Coxeter group. By definition, W acts on the vector space V . Once a basis for V is chosen, we obtain an action of W on the coordinates of V , and in turn the symmetric algebra $S(V) \cong \mathbb{C}[x^1, \dots, x^N]$ of polynomials in these coordinates. Inside $S(V)$ there sits a distinguished class of polynomial, $S(V)^W$, that are invariant under the action of the Coxeter group W . This subalgebra has a very nice basis:

Theorem 4 (Chevalley). *$S(V)^W$ is generated by N algebraically independent homogeneous polynomials,*

$$\mathbb{C}[x^1, \dots, x^N]^W \cong \mathbb{C}[y^1, \dots, y^N]. \quad (1.6.2)$$

Proof. Is beyond the scope of the present discussion; we refer the reader to Humphreys [36]. \square

The degrees of these invariant polynomials are uniquely determined by the Coxeter group. Writing $d_\alpha = \deg(y^\alpha)$, we arrange them such that

$$d_1 = h > d_2 \geq \dots \geq d_{N-1} > d_N = 2.$$

The sum

$$d_\alpha + d_{N-\alpha+1} = h + 2,$$

does not depend on α , and is known as the *duality condition*. The degree h of the highest weight polynomial is known as the Coxeter number. The strict inequality $h > d_2$ means that the polynomial y^1 , and so the vector field

$$e = \frac{\partial}{\partial y^1},$$

are fixed uniquely up to a scalar multiple. The existence of a degree 2 invariant for any Coxeter group is obvious: it is a function of distance from the origin. In particular, we can take

$$y^N = \frac{1}{2h} \sum_{i=1}^N (x^i)^2.$$

The vector field encoding the homogeneity of the basic invariant polynomials is

$$E = \frac{1}{h} \sum_{\alpha=1}^N d_{\alpha} y^{\alpha} \frac{\partial}{\partial y^{\alpha}}; \quad E(y^{\alpha}) = d_{\alpha} y^{\alpha}. \quad (1.6.3)$$

Example 15 (Dihedral Groups). *Consider a regular k -gon in $\mathbb{R}^2 \cong \mathbb{C}$, centered at the origin, and one of whose vertices lies on the x -axis (real axis). The group of symmetries of the k -gon, denoted $I_2(k)$, is generated by the rotation $z \mapsto e^{2\pi i/k} z$, and reflection $z \mapsto \bar{z}$. The basic invariant polynomials are*

$$\begin{aligned} y^1 &= z^k + \bar{z}^k, \\ y^2 &= \frac{1}{2k} z \bar{z}. \end{aligned}$$

Assuming that the original coordinates $\{x^i : i = 1, \dots, N\}$ are orthogonal with respect to the standard Euclidean metric on V , $g_{ij} = \delta_{ij}$, we have in the basis $\{y^{\alpha} : \alpha = 1, \dots, N\}$, of W -invariant polynomials

$$g^{\alpha\beta}(y) = \sum_{i=1}^N \frac{\partial y^{\alpha}}{\partial x^i} \frac{\partial y^{\beta}}{\partial x^i}. \quad (1.6.4)$$

The corresponding connection one-forms read

$$\sum_{\gamma=1}^N {}^g\Gamma_{\gamma}^{\alpha\beta}(y) dy^{\gamma} = \sum_{i,j,k=1}^N \frac{\partial^2 y^{\alpha}}{\partial x^i \partial x^j} \frac{\partial y^{\beta}}{\partial x^k} dx^k. \quad (1.6.5)$$

Proposition 3. [16] *The functions $g^{\alpha\beta}(y)$, $\Gamma_{\gamma}^{\alpha\beta}(y)$ depend at most linearly on y^1 .*

Proof. It follows from the definitions (1.6.4), (1.6.5) that the functions $g^{\alpha\beta}(y)$, $\Gamma_{\gamma}^{\alpha\beta}(y)$ are polynomials of degree

$$\begin{aligned} \deg(g^{\alpha\beta}(y)) &= d_{\alpha} + d_{\beta} - 2, \\ \deg({}^g\Gamma_{\gamma}^{\alpha\beta}(y)) &= d_{\alpha} + d_{\beta} - d_{\gamma} - 2. \end{aligned}$$

Owing to the ordering of the degrees, $d_{\alpha} + d_{\beta} \leq 2d_1 = 2h$, these polynomials can be at most linear in y^1 . \square

This Proposition is key in constructing the Frobenius manifold structure. We define

$$\eta^{\alpha\beta}(y) := \frac{\partial}{\partial y^1} g^{\alpha\beta}(y), \quad (1.6.6)$$

$$\eta_{\gamma}^{\alpha\beta}(y) := \frac{\partial}{\partial y^1} {}^g\Gamma_{\gamma}^{\alpha\beta}(y). \quad (1.6.7)$$

The bilinear form (1.6.6) is called the *Saito* metric after Kyoji Saito. That we are calling

this bilinear form a metric is justified by Lemma 5. Further, the Lemma tells us that the pair of metrics η and g form a flat pencil. This flat pencil is regular and quasihomogeneous with respect to the vectors fields e and E defined above, and so by Theorem 3 a unique, up to isomorphism, Frobenius manifold structure exists on \mathcal{M} .

Let us follow the recipe outlined by Theorem 3 to construct the Frobenius manifold structure on the orbit space of the dihedral group $I_2(k)$.

Example 16 (Dihedral Groups Continued). *The standard Euclidean (contravariant) metric on $\mathbb{R}^2 \cong \mathbb{C}$ is given by*

$$g = 2 \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}} + 2 \frac{\partial}{\partial \bar{z}} \otimes \frac{\partial}{\partial z}.$$

In the system of invariant polynomials, we have

$$g = 2k^{k+1}y_2^{k-1} \frac{\partial}{\partial y_1} \otimes \frac{\partial}{\partial y_1} + 2y_1 \frac{\partial}{\partial y_1} \otimes \frac{\partial}{\partial y_2} + \frac{2}{k}y_2 \frac{\partial}{\partial y_2} \otimes \frac{\partial}{\partial y_2}.$$

We can solve the system (1.4.29), (1.4.30) to obtain the non-zero Christoffel symbols,

$$\Gamma_2^{11} = k^{k+1}(k-1)y_2^{k-2}, \quad \Gamma_1^{12} = \frac{k-1}{k}, \quad \Gamma_2^{22} = \frac{1}{k}, \quad \Gamma_1^{21} = \frac{1}{k}.$$

Equation (1.6.3) gives the Euler field

$$E = y_1 \frac{\partial}{\partial y_1} + \frac{1}{2k}y_2 \frac{\partial}{\partial y_2},$$

while the unity is given by

$$e = \frac{\partial}{\partial y_1}.$$

The tensor R defined by (1.4.31) is invertible, with inverse

$$R^{-1} = \frac{k}{k-1} \frac{\partial}{\partial y_1} \otimes dy_1 + k \frac{\partial}{\partial y_2} \otimes dy_2.$$

Using this we obtain the multiplication table

$$\begin{aligned} \frac{\partial}{\partial y_1} \circ \frac{\partial}{\partial y_1} &= \frac{\partial}{\partial y_1}, \\ \frac{\partial}{\partial y_1} \circ \frac{\partial}{\partial y_2} &= \frac{\partial}{\partial y_2}, \\ \frac{\partial}{\partial y_2} \circ \frac{\partial}{\partial y_2} &= k^{k+2}(k-1)y_2^{k-2} \frac{\partial}{\partial y_1}, \end{aligned}$$

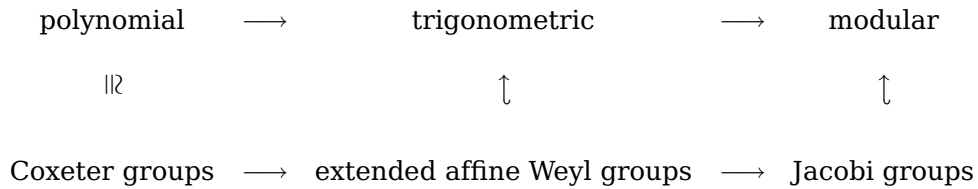
which gives

$$F = \frac{1}{2}y_1^2y_2 + \frac{k^{k+1}}{k+1}y_2^{k+1}.$$

This is solution (1.2.13), up to a re-scaling of the functions y_1, y_2 .

It is clear from the construction that any solution to the WDVV equations arising in this way will be polynomial - the components of the intersection form will be polynomials in the basis of invariant polynomials of the Coxeter group. This polynomiality is then inherited by the Christoffel symbols and Saito metric, from which the structure constants of the algebra are built (without taking quotients). After the details of this construction were given by Dubrovin, a beautiful result of Hertling [34] states that *every* polynomial Frobenius manifold arises in this way. Thus the orbit space construction sketched here provides a classification of polynomial solutions to the WDVV equations.

Beyond polynomial solutions to WDVV, there are solutions that involve trigonometric and modular functions. Analogously, beyond the Coxeter groups, there are extended affine Weyl groups and so-called Jacobi groups, whose rings of invariants are no longer polynomials, but instead trigonometric and modular functions respectively. This perhaps suggests that progress towards classification of trigonometric and modular solutions to the WDVV equations may arise through the study of Frobenius structures on the orbit spaces of these groups. More specifically, Dubrovin and Zhang [22] provided the orbit space construction for the case of extended affine Weyl groups, and in his Ph.D. thesis, Bertola [6] showed how to construct a Frobenius manifold on the orbit spaces of a Jacobi group, one example of which is the solution presented in Example 5 [16]. The problem of classifying solutions to the WDVV equations is an open problem, the current state of play being summarised most succinctly in a diagram:



The solutions found by Bertola, together with the solution found by Verlinde and Warner [66], fall into a class of Frobenius manifolds that we will call *modular Frobenius manifolds* - they are examples of objects in the codomain of the rightmost map. In the next chapter we will set out a program for how one might tackle the problem from the side of the WDVV equations, and provide classification results for low dimensions. We will show that all of the solutions we find do not violate the possibility that the rightmost map in the diagram is also surjective.

Chapter 2

Modular Frobenius Manifolds

2.1 Modular Frobenius Manifolds

The inversion symmetry singles out a special class of Frobenius manifolds: those that lie at its fixed points. Such Frobenius manifolds have a particularly rich structure. As we will see, the prepotential will be a quasi-modular form. It is interesting that some of the first solutions (1.5.1) of the WDVV equations found by Verlinde & Warner [66] were of this type. More recently prepotentials of this form are appearing in the study of orbifold quantum cohomology [46, 59, 11]. As mentioned at the end of the previous chapter, the examples of Bertola [6] coming from the orbit spaces of Jacobi groups also fall into this class.

Such solutions are isolated examples of *modular Frobenius manifolds*. We will demonstrate how demanding invariance of the prepotential under the inversion symmetry restricts its functional form, and set out a framework for classification of prepotentials with this property. We give classification results for so-called *polynomial modular Frobenius manifolds* in dimensions 3 and 4, with partial results for dimension 5.

To motivate what follows, let us consider an example (taken from [16] Appendix C).

Example 17. [Continued from Example 5, page 12] Recall that under the inversion symmetry, the functional form of the Frobenius manifold defined by the data

$$F(t^1, t^2, t^3) = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2}t^1(t^2)^2 - \frac{(t^2)^4}{16}\gamma(t^3), \quad E = t^1 \frac{\partial}{\partial t^1} + \frac{1}{2}t^2 \frac{\partial}{\partial t^2}. \quad (2.1.1)$$

was preserved, and the WDVV equations are equivalent to the Chazy equation,

$$\gamma''' = 6\gamma\gamma'' - 9(\gamma')^2 \quad (2.1.2)$$

for the function $\gamma(t^3)$. One simple explicit solution of the Chazy equation is given by

$$\gamma(t^3) = \frac{\pi i}{3} E_2(t^3),$$

where E_2 is the second Eisenstein series,

$$E_2(\tau) = \frac{1}{3\pi^2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(m + n\tau)^2}.$$

With this explicit solution one has a stronger symmetry. Since E_2 has the modularity properties

$$E_2(\tau + 1) = E_2(\tau), \quad E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{12}{2\pi i} \tau \quad (2.1.3)$$

then under inversion symmetry one has $\hat{F}(\hat{t}) = F(\hat{t})$, i.e. both the functional form of the prepotential and the specific solution of the Chazy equation are preserved.

We would like to study the class of Frobenius manifold that this example falls into. The characterising property of this class is an invariance for the modular group of $SL(2, \mathbb{Z})$. The inversion symmetry defines a representation of the generator $\tau \mapsto -1/\tau$ on the space of Frobenius manifolds. A modular Frobenius manifold must also be invariant under periodic shifts in τ :

Definition 9. Let \mathcal{M} be a Frobenius manifold with fixed coordinates $\{\mathbf{t}\}$, and Euler vector field E . The manifold \mathcal{M} is defined to be a modular Frobenius manifold if both

1. The prepotential is invariant under periodic shifts in t^N ,

$$F(t^1, \dots, t^{N-1}, t^N + 1) = F(t^1, \dots, t^{N-1}, t^N) \quad \text{modulo quadratic terms.} \quad (2.1.4)$$

2. The data \hat{F} , \hat{E} , $\{\hat{\mathbf{t}}\}$ defining the image $\widehat{\mathcal{M}}$ of the Frobenius manifold \mathcal{M} under the inversion symmetry, is given by

$$\hat{F}(\hat{\mathbf{t}}) = F(\hat{\mathbf{t}}), \quad \text{and} \quad \hat{E}(\hat{\mathbf{t}}) = E(\hat{\mathbf{t}}). \quad (2.1.5)$$

Thus instead of transforming the original prepotential into a different prepotential, the inversion symmetry maps the prepotential to itself, as in Example 5, and hence one may think of modular Frobenius manifolds as lying at the fixed point of the involutive symmetry I . A comparison of the two Euler fields, using (1.5.6) and (1.5.7), gives the following constraints: $\hat{d} = d$ and hence $d = 1$. Thus we have the following necessary conditions for a Frobenius manifold to be modular:

$$E = \sum_{\alpha=1}^{N-1} \left(\frac{1}{2} - \mu_\alpha\right) t^\alpha \partial_\alpha, \quad (2.1.6)$$

$$\mathcal{L}_E F = 2F. \quad (2.1.7)$$

Since the variables t^1 and t^N behave differently from the remaining variables it is useful to use a different notation, namely:

$$u = t^1, \quad z^i = t^i, i = 2, \dots, N-1, \quad \tau = t^N,$$

and $\mathbf{z} = (z^2, \dots, z^{N-1})$. The two notations will be used interchangeably in what follows.

Proposition 4. [16] The group $SL(2, \mathbb{C})$ acts on the space of modular Frobenius manifolds

by

$$\begin{aligned} u &\mapsto \hat{u} = u + \frac{1}{2} \frac{c}{c\tau + d} \eta(\mathbf{z}, \mathbf{z}), \\ \mathbf{z} &\mapsto \hat{\mathbf{z}} = \frac{\mathbf{z}}{c\tau + d}, \\ \tau &\mapsto \hat{\tau} = \frac{a\tau + b}{c\tau + d}, \end{aligned} \tag{2.1.8}$$

for $ad - bc = 1$.

Proof. Note that the re-scalings $\tau \mapsto a\tau + b$ act trivially on the solution space to WDVV with $d = 1$ ($\partial/\partial\tau \in \text{Ker}Q$): they leave the WDVV equations and scaling condition $\mathcal{L}_E F = 2F$ invariant. Composing these re-scalings with the invariance (2.1.5) of F under the inversion symmetry, we arrive at the desired action. \square

A simple calculation then yields the modularity properties of the non-cubic part of the prepotential.

Proposition 5. *The prepotential*

$$F = \frac{1}{2} u^2 \tau - \frac{1}{2} u \eta(\mathbf{z}, \mathbf{z}) + f(\mathbf{z}, \tau) \tag{2.1.9}$$

(where $\eta(W, \mathbf{y}) = \sum_{i,j=2}^{N-1} \eta_{ij} x^i y^j$) defines a modular Frobenius manifold if and only if

$$f\left(\frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right) = \frac{1}{\tau^2} f(\mathbf{z}, \tau) - \frac{1}{4\tau^3} \eta(\mathbf{z}, \mathbf{z})^2, \quad \text{and} \quad f(\mathbf{z}, \tau + 1) = f(\mathbf{z}, \tau) \tag{2.1.10}$$

2.2 Modularity and Quasi-Homogeneity

We will consider polynomial prepotentials of the form (2.1.9) with

$$f = \sum_{\alpha \in \mathbb{L}} \left\{ \prod_{i=2}^{N-1} (t^i)^{\alpha_i} \right\} g_{\alpha}(t^N), \tag{2.2.1}$$

so $\mathbf{z} = \{t^2, \dots, t^{N-1}\}$, $\tau = t^N$ and $\alpha = (\alpha_2, \dots, \alpha_{N-1}) \in \mathbb{L} := \mathbb{Z}_{\geq 0}^{N-2}$. We do not assume any general properties of the functions g_{α} at this stage - the aim is to obtain the differential equations that these functions must satisfy in order for the WDVV equations to hold. Boundary or other conditions (for example, solutions being analytic at infinity) will then place constraints on these functions, but a priori no such constraints will be imposed. We will also impose the assumption of semi-simplicity on the solutions we seek. As we will see, this then places extremely strong restrictions on the possible values of the weights d_i . The following definition, while not immediately obvious, will play an important role in what follows.

Definition 10. A pivot-point $\alpha \in \mathbb{L}$ is a lattice point for which

$$\text{coeff}_\alpha \left(\sum_{i,j=2}^{N-1} \eta_{ij} t^i t^j \right)^2 \neq 0,$$

where $\text{coeff}_\alpha(p)$ is the coefficient of $\prod_{i=2}^{N-1} (t^i)^{\alpha_i}$ of the polynomial p .

Proposition 6. Assume that f defined above satisfies the modularity condition (2.1.10). Then

$$g_\alpha(\tau + 1) = g_\alpha(\tau),$$

and:

- If α is not a pivot-point then:

$$g_\alpha \left(-\frac{1}{\tau} \right) = \tau^{(\sum \alpha_i) - 2} g_\alpha(\tau);$$

- If α is a pivot-point then:

$$g_\alpha \left(-\frac{1}{\tau} \right) = \tau^2 g_\alpha(\tau) - \frac{1}{4} \tau \text{coeff}_\alpha \left(\sum_{i,j=2}^{N-1} \eta_{ij} t^i t^j \right)^2$$

The proof is by direct computation and will be omitted.

These modularity properties place constraints on the functions lying at pivot points, illustrated by the following

Lemma 6. Let $g : \mathbb{H} \rightarrow \mathbb{C} \cup \{\infty\}$ be a meromorphic function enjoying the transformation properties

$$g \left(\frac{-1}{\tau} \right) = \tau^2 g(\tau) + a\tau, \quad g(\tau + 1) = g(\tau) \tag{2.2.2}$$

for some constant a . Then

$$g(\tau) = a \frac{\pi i}{6} E_2(\tau) + \frac{d}{d\tau} (P(j(\tau))), \tag{2.2.3}$$

for some polynomial P in the modular j -invariant.

Proof. Recall that the j -invariant has a simple pole at $q = 0$ [50]:

$$j(\tau) \sim \frac{1}{q} + \text{positive powers of } q, \quad q = e^{2\pi i \tau}$$

Also, note that

$$\left(g - \frac{d}{d\tau} P(j) \right) \left(\frac{-1}{\tau} \right) = \tau^2 \left(g - \frac{d}{d\tau} P(j) \right) (\tau) + \frac{\pi i}{6} \tau,$$

We can define a polynomial P such that

$$\left(g - \frac{d}{d\tau}(P(j))\right)(\tau) = a_0 + \sum_{n>0} a_n q^n,$$

i.e. contains only positive powers of q . This means that $g - \frac{d}{d\tau}P(j)$ is a quasi-modular form of weight 2, i.e. proportional to E_2 . \square

Note that without placing analytic constraints on the function g , we cannot fix the polynomial P .

Having determined the modularity properties of the functions we now apply quasi-homogeneity. This will determine the possible terms that can appear in the ansatz (2.2.1). Recall that the Euler field takes the form

$$E = \sum_{i=1}^{N-1} d_i t^i \frac{\partial}{\partial t^i}$$

where $d_i + d_{N+1-i} = 1$. We now assume further that the d_i are positive rational numbers. Applying the quasihomogeneity condition $E(F) = 2F$ (recall modular Frobenius manifold must have $d = 1$) implies the following constraint on the α_i :

$$\sum_{i=2}^{N-1} d_i \alpha_i = 2. \quad (2.2.4)$$

Thus given the d_i we arrive at a special Diophantine equation whose solutions determines the possible monomials in (2.2.1) (rather remarkably, this special type of Diophantine equation is known in number theory as a Frobenius equation).

Pivot-points - which are defined without reference to the Euler vector field - play an important role in the construction of solutions of this equation.

Lemma 7. *Pivot-points automatically satisfy the Frobenius equation (2.2.4).*

Proof. On expanding $\left(\sum_{i,j=2}^{N-1} \eta_{ij} t^i t^j\right)^2$ one can obtain the form of the pivot-points, namely (and here only non-zero elements are shown, all other elements are zero):

- (i) $\alpha = (\dots, 2, \dots, 2, \dots)$ with 2 in the i and $N+1-i$ positions;
- (ii) $\alpha = (\dots, 1, \dots, 1, \dots, 1, \dots, 1, \dots)$ with 1 in the i, j and $N+1-i, N+1-j$ positions ($i \neq j$).

If N is odd one obtains two further pivot-points:

- (iii) $\alpha = (\dots, 4, \dots)$ with 4 in the middle position;
- (iv) $\alpha = (\dots, 1, \dots, 2, \dots, 1, \dots)$ with 1 in the i and $N+1-i$ positions and 2 in the middle position.

Since $d_i + d_{N+1-i} = 1$ (and hence $d_{(N+1)/2} = 1/2$ if N is odd) the result follows.

Given the above forms of the pivot-points one can easily count the number of such points: if N is even the number of pivot-points is $N(N-2)/8$ and if N is odd the number of pivot-points is $(N+1)(N-1)/8$.

□

Example 18. Suppose $N = 8$. Then

$$\left(\sum_{i,j=2}^{N-1} \eta_{ij} t^i t^j \right)^2 = (t_2 t_7 + t_3 t_6 + t_4 t_5)^2 = (t_2 t_7)^2 + (t_3 t_6)^2 + (t_4 t_5)^2 + 2t_2 t_7 t_3 t_6 + 2t_3 t_6 t_4 t_5 + 2t_2 t_7 t_4 t_5,$$

and we have six pivot points

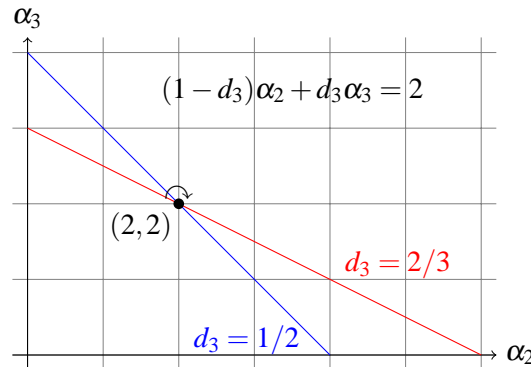
$$\{(2, 0, 0, 0, 0, 2), (0, 2, 0, 0, 2, 0), (0, 0, 2, 2, 0, 0), \\ (1, 1, 0, 0, 1, 1), (0, 1, 1, 1, 1, 0), (1, 0, 1, 1, 0, 1)\}.$$

Note that the first three are of the form (i) and the latter form (ii). They all satisfy the Frobenius equation. For example, consider the first one:

$$2d_2 + 2d_7 = 2(d_2 + d_7) = 2,$$

since $d_i + d_{N-i+1} = 1$.

One can now give the geometric motivation for the name ‘pivot-point’. The Frobenius equation (2.2.4) defines a hyperplane Π in α -space and we wish to find integer solutions, i.e. the points in $\Pi \cap \mathbb{L}$. Since pivot-points are independent of the d_i as the d_i vary the plane Π ‘pivots’ around these points. In the simplest non-trivial example, when $N = 4$, we have $(1 - d_3)\alpha_2 + d_3\alpha_3 = 2$ and a single pivot-point $(2, 2)$. Thus as d_3 varies the line rotates, or pivots, about this point:



It is also obvious geometrically from the direction of the normal vector to Π that the number of solutions to the Frobenius equation is finite and hence (2.2.1) is a polynomial in the variables $\{t^2, \dots, t^{N-1}\}$.

The number of independent pivot terms can be reduced further; in fact to one. Let $\gamma(\tau)$ be any function with the transformation property

$$\gamma\left(-\frac{1}{\tau}\right) = \tau^2 \gamma(\tau) - \frac{1}{4} \tau$$

and let

$$f(\mathbf{z}, \tau) = \gamma(\tau)(\mathbf{z}, \mathbf{z})^2 + g(\mathbf{z}, \tau).$$

Then equation (2.1.10) implies that

$$g\left(\frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right) = \frac{1}{\tau^2} g(\mathbf{z}, \tau).$$

Thus one can obtain a refinement of Proposition 5:

Proposition 7. *The prepotential of a modular Frobenius manifold takes the form*

$$F = \frac{1}{2} u^2 \tau - \frac{1}{2} u(\mathbf{z}, \mathbf{z}) + \gamma(\tau)(\mathbf{z}, \mathbf{z})^2 + g(\mathbf{z}, \tau) \quad (2.2.5)$$

where

$$\gamma\left(-\frac{1}{\tau}\right) = \tau^2 \gamma(\tau) - \frac{1}{4} \tau, \quad \gamma(\tau+1) = \gamma(\tau), \quad g\left(\frac{\mathbf{z}}{\tau}, -\frac{1}{\tau}\right) = \frac{1}{\tau^2} g(\mathbf{z}, \tau), \quad g(\tau+1) = g(\tau).$$

Moreover, if

$$g(\mathbf{z}, \tau) = \sum_{\alpha \in \mathbb{L} \cap \Pi} \left\{ \prod_{i=2}^{N-1} (z^i)^{\alpha_i} \right\} g_{\alpha}(\tau)$$

then

$$g_{\alpha}\left(-\frac{1}{\tau}\right) = \tau^{(\sum \alpha_i) - 2} g_{\alpha}(\tau).$$

Thus modularity and quasi-homogeneity determine the modularity properties of the functions γ and g_{α} together with the form of the monomial coefficients of these functions. We have arrived at the most concise ansatz possible given our assumptions. The next step will be to posit our ansatz as a solution to the WDVV equations, which will turn out to be equivalent to ordinary differential equations in the functions $\{\gamma, g_{\alpha}\}$, with a modular symmetry of the same flavour as that of the Chazy equation (2.1.2). When we have found these ordinary differential equations we can check, without first having to solve them, whether or not the resulting solution to WDVV will be semi-simple.

Remark. Another functional class in which to look for modular manifolds could be obtained by weakening the polynomial ansatz and replacing it with a rational ansatz (e.g rational in the variable t^{N-1} and polynomial in the variables t^1, \dots, t^{N-2}) - this would then include the A_{N-2} and B_{N-2} examples of Bertola [6]. These examples, while rational, are

constrained by the condition that the functions $g_\alpha(\tau)$ are never of negative weight, i.e. $\sum \alpha_i \geq 2$. Such a generalization will not be pursued here.

2.3 The WDVV Equations and Modular Dynamical Systems

Recall that the WDVV equations are satisfied if and only if all of our algebras are associative. Therefore in order to analyze the equations, we consider the obstruction to associativity

$$\Delta[X, Y, Z] = (X \circ Y) \circ Z - X \circ (Y \circ Z).$$

We are assuming the algebras are unital, and so if any of the vector fields X, Y , or Z are equal to the unity field then Δ vanishes identically. The form of our ansatz means that the variable t^n (also denoted τ) is also special. For example, the dependence of our ansatz for the prepotential on the variable τ is *not* polynomial. The variable τ also behaves differently under the inversion symmetry. Therefore we decompose the WDVV equations into different classes, determined by the number of τ -derivatives present. Taking the inner product of $\Delta[X, Y, Z]$ with a fourth arbitrary vector field W in order to obtain scalar valued equations, we arrive at the following:

Proposition 8. *The WDVV equations for a multiplication with unity field are equivalent to the vanishing of the following functions:*

$$\Delta^{(1)}(X, Y) = \eta(\partial_\tau \circ \partial_\tau, X \circ Y) - \eta(\partial_\tau \circ X, \partial_\tau \circ Y),$$

$$\Delta^{(2)}(X, Y, Z) = \eta(\partial_\tau \circ X, Y \circ Z) - \eta(\partial_\tau \circ Z, X \circ Y),$$

$$\Delta^{(3)}(X, Y, Z, W) = \eta(X \circ Y, Z \circ W) - \eta(X \circ W, Y \circ Z)$$

for all $X, Y, Z, W \in \text{span}\{ \partial_{t^i} : i = 2, \dots, N-1 \}$.

In terms of coordinate vector fields these conditions are:

$$\begin{aligned} \Delta_{ij}^{(1)} &= \eta_{ij} c_{\tau\tau\tau} + \eta^{pq} \{ c_{\tau\tau p} c_{ijq} - c_{\tau ip} c_{\tau jq} \}, \\ \Delta_{ijk}^{(2)} &= \{ \eta_{jk} c_{\tau\tau i} - \eta_{ij} c_{\tau\tau k} \} + \eta^{pq} \{ c_{\tau ip} c_{jkq} - c_{\tau kp} c_{ijq} \} \\ \Delta_{ijrs}^{(3)} &= \{ \eta_{ij} c_{\tau rs} + \eta_{rs} c_{\tau ij} - \eta_{is} c_{\tau rj} - \eta_{rj} c_{\tau is} \} + \eta^{pq} \{ c_{ijp} c_{rsq} - c_{isp} c_{rjq} \}. \end{aligned} \tag{2.3.1}$$

Hence imposing the WDVV equations gives systems of over-determined non-linear ordinary differential equations (in the variable τ). By construction, these systems will possess similar properties to the Chazy equation (2.1.2): their solutions will have an $SL(2, \mathbb{C})$ symmetry. To get a modular Frobenius manifold one then looks for solutions

which have an invariance under $SL(2, \mathbb{Z})$. This was illustrated in Example 5, when we chose the solution

$$\gamma(\tau) = \frac{\pi i}{6} E_2(\tau)$$

to (2.1.2). In addition, if one then imposes the condition of semi-simplicity on the resulting multiplication one finds that this then leads to very strong constraints on the possible d_i that give modular Frobenius manifolds.

The transformation properties of the functions $\{\gamma, g_\alpha\}$ derived above, together with the equations (2.3.1) that they must satisfy motivate the following. Suppose that one has a set of quasi-modular functions (we assume no analytic properties such as being holomorphic in the upper-half plane) $\gamma(\tau)$ and $g_\alpha(\tau)$, $\alpha \in \mathcal{W}$ for some indexing set \mathcal{W} such that:

$$\begin{aligned} \gamma\left(-\frac{1}{\tau}\right) &= \tau^2 \gamma(\tau) + a\tau, & a \neq 0 \text{ constant}, \\ g_n\left(-\frac{1}{\tau}\right) &= \tau^n g_n(\tau). \end{aligned}$$

Such a g_n is said to have weight n . Define next a Rankin-type derivative

$$\begin{aligned} D(g_n) &= \frac{dg_n}{d\tau} - \frac{n\gamma}{a} g_n, \\ D(\gamma) &= \frac{d\gamma}{d\tau} - \frac{1}{a} \gamma^2. \end{aligned}$$

It is easy to check that $D(\gamma)$ has weight 4 and $D(g_n)$ has weight $n+2$ and that $D(g_n g_m) = g_n Dg_m + g_m Dg_n$.

Definition 11. Let \mathcal{W} be some (finite) indexing set. A modular dynamical system takes the form

$$\begin{aligned} D(\gamma) &= q(\mathbf{g}), \\ D(g_\alpha) &= p_\alpha(\mathbf{g}), & \alpha \in \mathcal{W}. \end{aligned}$$

where the polynomials q and p_α are constrained so that the resulting system is invariant under the transformation $\tau \rightarrow -\frac{1}{\tau}$.

Example 19 (The Ramanujan System). The Eisenstein series are defined by

$$E_{2k}(\tau) = \frac{1}{2\zeta(2k)} \sum_{(n,m) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(m+n\tau)^{2k}},$$

where $\zeta(s)$ is Riemann's zeta function, $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$. From the definition it follows that

for $k \geq 2$

$$\begin{aligned} E_{2k}(\tau+1) &= E_{2k}(\tau), \\ E_{2k}\left(-\frac{1}{\tau}\right) &= \tau^{2k} E_{2k}(\tau). \end{aligned}$$

If $k = 1$, we have

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{12}{2\pi i} \tau,$$

as above. Srinivasa Ramanujan found the following modular dynamical system on $\{E_2, E_4, E_6\}$:

$$\begin{aligned} E_2' &= \frac{1}{12}(E_2^2 - E_4), \\ E_4' &= \frac{1}{3}(E_2 E_4 - E_6), \\ E_6' &= \frac{1}{2}(E_2 E_6 - E_4^2), \end{aligned}$$

where $' \equiv q \frac{d}{dq}$; $q = e^{2\pi i \tau}$. Here E_2 plays the role of γ , defining the Rankin derivative.

Proposition 9. *The WDVV equations for a polynomial modular Frobenius manifold are equivalent to a modular dynamical system.*

Proof. We saw in Proposition 7 that the number of pivot functions present in the prepotential of a modular Frobenius manifold may be reduced to one. Hence the obstructions to WDVV take the form

$$\begin{aligned} \Delta^{(3)} = 0 &\Rightarrow \begin{cases} D\gamma &= p_\gamma(g_1, \dots, g_k), \\ Dg_i &= p_i(g_1, \dots, g_k), \quad \text{for } i = 1, \dots, k, \end{cases} \\ \Delta^{(2)} = 0 &\Rightarrow \begin{cases} D^2\gamma &= q_\gamma(g_1, \dots, g_k), \\ D^2g_i &= q_i(g_1, \dots, g_k), \quad \text{for } i = 1, \dots, k, \end{cases} \\ \Delta^{(1)} = 0 &\Rightarrow \begin{cases} D^3\gamma &= r_\gamma(g_1, \dots, g_k), \\ D^3g_i &= r_i(g_1, \dots, g_k), \quad \text{for } i = 1, \dots, k, \end{cases} \end{aligned}$$

where k is the number of non-pivot functions present in the ansatz (2.2.5), and $\{p_\sigma, q_\sigma, r_\sigma : \sigma = 1, \dots, k, \gamma\}$ are polynomials of various degrees such that the resulting system is modular invariant. \square

2.3.1 Modular Dynamical Systems for $N = 3$.

In this simplest case (see Example 5) there is no freedom: the Euler vector field is fixed by the fact that $d = 1$ and hence

$$E = t^1 \frac{\partial}{\partial t^1} + \frac{1}{2} t^2 \frac{\partial}{\partial t^2}$$

The Frobenius equation (2.2.4) only has one solution and hence modularity and quasi-homogeneity imply a prepotential of the form

$$F = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 - \frac{1}{16} t_2^4 \gamma(t_3).$$

The WDVV equations then imply that γ must satisfy the third order equation

$$\gamma''' - 6\gamma\gamma'' + 9(\gamma')^2 = 0. \quad (2.3.2)$$

This is nothing more than the Chazy equation, whose modularity properties are well known (see [16]). Under the identification

$$\gamma(\tau) = -\frac{2}{3}(\omega_1(\tau) + \omega_2(\tau) + \omega_3(\tau)),$$

the Chazy equation is equivalent to the Halphen system (2.3.3):

$$\begin{aligned} \dot{\omega}_1 &= -\omega_2\omega_3 + \omega_1(\omega_2 + \omega_3), \\ \dot{\omega}_2 &= -\omega_3\omega_1 + \omega_2(\omega_3 + \omega_1), \\ \dot{\omega}_3 &= -\omega_1\omega_2 + \omega_3(\omega_1 + \omega_2), \end{aligned} \quad \cdot \equiv \frac{d}{d\tau}$$

on the functions $\{\omega_1, \omega_2, \omega_3\}$. Here the $SL(2, \mathbb{C})$ action is defined by

$$\tau \mapsto \hat{\tau} = \frac{a\tau + b}{c\tau + d}, \quad \omega_i(\tau) \mapsto \hat{\omega}_i(\hat{\tau}) = (c\tau + d)^2 \omega_i\left(\frac{a\tau + b}{c\tau + d}\right) - (c\tau + d), \quad \text{for } i = 1, 2, 3.$$

In order to construct a modular Frobenius manifold, one must now choose an $SL(2, \mathbb{Z})$ -invariant solution to the modular dynamical system. One can also use the techniques developed in the next section to solve the Chazy equation (2.3.2) in terms of the Schwarzian triangle function $S[\frac{1}{2}, \frac{1}{3}, 0, t]$.

2.3.2 Modular Dynamical Systems for $N = 4$.

In this case there is a 1-parameter family of possible Euler fields, namely

$$E = t^1 \frac{\partial}{\partial t^1} + (1 - \sigma) t^2 \frac{\partial}{\partial t^2} + \sigma t^3 \frac{\partial}{\partial t^3}.$$

Without loss of generality we may assume decreasing degrees and hence $\sigma \leq \frac{1}{2}$. A detailed analysis of the Frobenius equation (2.2.4) gives the solutions summarized below:

σ	Non-pivot solutions (α_2, α_3) :
$\frac{1}{2}$	$(4, 0), (3, 1), (1, 3), (0, 4);$
$\frac{1}{3}$	$(3, 0), (1, 4), (0, 6);$
$\frac{1}{n}, (n \geq 4)$	$(1, n+1), (0, 2n);$
$\frac{2}{2n+1}, (n \geq 2)$	$(0, 2n+1);$
other	none.

Table 2.1: Non-pivot solutions of the Frobenius equation (2.2.4) for $N = 4$.

We now analyze each of the four cases in turn.

$N = 4, \sigma = 1/n, n \geq 4$.

We have the ansatz

$$\begin{aligned}
 F &= \frac{1}{2}t_1^2t_4 + t_1t_2t_3 - \frac{1}{4}\gamma(t^4)(t_2t_3)^2 + g_1(t^4)t_2t_3^{n+1} + g_2(t_4)t_3^{2n}; \\
 E &= t_1\frac{\partial}{\partial t_1} + \frac{n-1}{n}t_2\frac{\partial}{\partial t_2} + \frac{1}{n}t_3\frac{\partial}{\partial t_3}.
 \end{aligned}$$

Where

$$g_1\left(\frac{-1}{\tau}\right) = \tau^n g(\tau), \quad g_2\left(\frac{-1}{\tau}\right) = \tau^{2n-2} g_2(\tau), \quad \gamma\left(\frac{-1}{\tau}\right) = \tau^2 \gamma(\tau) + 2\tau.$$

Requiring $\Delta^{(3)} = 0$ means that the function γ must satisfy a first order differential equation of the form

$$D\gamma = p(g_1, g_2)$$

where p is a polynomial in the functions g_1 and g_2 which makes the differential equation invariant under $\tau \mapsto -1/\tau$. Note that, by definition,

$$(D\gamma)\left(\frac{-1}{\tau}\right) = \tau^4 D\gamma(\tau).$$

But the weights of g_1, g_2 are such that there does *not* exist a polynomial satisfying

$$p(g_1, g_2)\left(\frac{-1}{\tau}\right) = \tau^4 p(g_1, g_2)(\tau).$$

Therefore

$$D\gamma = 0.$$

Recalling the definition of the operator D , this is

$$\gamma' = \frac{1}{2}\gamma^2 \Rightarrow \gamma(\tau) = \frac{2}{\tau_0 - \tau}, \text{ for some constant } \tau_0.$$

This function does not satisfy the required transformation properties of a pivot function: it may be reduced to zero by the transformation

$$\tau \mapsto \hat{\tau} = \frac{1}{\tau - \tau_0}.$$

Hence any resulting solution to the WDVV equations lies on the same $SL(2, \mathbb{Z})$ orbit as the trivial 4 dimensional Frobenius manifold,

$$F = \frac{1}{2}t_1^2 t_4 + t_1 t_2 t_3,$$

which is not semi-simple.

$\mathbf{N} = 4, \sigma = 2/(2\mathbf{n} + 1), \mathbf{n} \geq 2$

This case is similar. We have the ansatz

$$\begin{aligned} F &= \frac{1}{2}t_1^2 t_4 + t_1 t_2 t_3 - \frac{1}{4}\gamma(t_4)(t_2 t_3)^2 + g(t_4)t_3^{2n+1}; \\ E &= t_1 \frac{\partial}{\partial t_1} + \frac{2n-1}{2n+1}t_2 \frac{\partial}{\partial t_2} + \frac{2}{2n+1}t_3 \frac{\partial}{\partial t_3}. \end{aligned}$$

The unknown functions $\{\gamma, g\}$ have the modularity properties

$$g\left(\frac{-1}{\tau}\right) = \tau^{2n-1}g(\tau), \quad \gamma\left(\frac{-1}{\tau}\right) = \tau^2\gamma(\tau) + 2\tau.$$

Vanishing of the obstruction $\Delta^{(3)}$ means γ must satisfy a first order differential equation. No positive integer power of the function g is a modular function of weight 4, so

$$D\gamma = 0,$$

and we arrive in the same situation as the previous case.

$\mathbf{N} = 4, \sigma = \frac{1}{2}$.

In this case we have the ansatz

$$\begin{aligned} F &= \frac{1}{2}t_1^2 t_4 + t_1 t_2 t_3 - \frac{1}{4}(t_2 t_3)^2 \gamma(t_4) + t_3^4 g_1(t_4) + t_2 t_3^3 g_2(t_4) + t_2^3 t_3 g_3(t_4) + t_2^4 g_4(t_4); \\ E &= t_1 \frac{\partial}{\partial t_1} + \frac{1}{2}t_2 \frac{\partial}{\partial t_2} + \frac{1}{2}t_3 \frac{\partial}{\partial t_3}. \end{aligned}$$

In this case the analysis is a lot more involved, and is typical of that required throughout

the rest of this section. For this reason we will give a lot of detail here, but will be more succinct in the analysis of further cases. We begin by first eliminating the obstruction $\Delta^{(3)}$. In particular, $\Delta_{3233}^{(3)} = 0 \Leftrightarrow$

$$\begin{aligned} g_2' &= \gamma g_2 + 24g_1g_3, & Dg_2 &= 24g_1g_3, \\ g_3' &= \gamma g_3 + 24g_2g_4, & Dg_3 &= 24g_2g_4, \\ \gamma' &= \frac{1}{2}\gamma^2 - 288g_1g_4, & D\gamma &= -288g_1g_4. \end{aligned} \quad (2.3.3)$$

Using these differential equations to replace all the first order derivatives of g_2, g_3 , and γ appearing in the WDVV equations we find that the obstruction $\Delta_{232}^{(2)} = 0 \Leftrightarrow$

$$24g_1g_3g_4 + g_2g_4\gamma - g_2g_4' = 0, \quad (2.3.4)$$

$$24g_1g_2g_4 + g_1g_3\gamma - g_3g_1' = 0, \quad (2.3.5)$$

$$g_4g_1' - g_1g_4' + 6(g_1g_3^2 - g_2^2g_4) = 0, \quad (2.3.6)$$

$$-3456g_1^2g_4 + 6g_1\gamma - 12\gamma g_1' + 4g_1'' = 0, \quad (2.3.7)$$

where we have suppressed the functional dependence on the variable t_4 , $' \equiv \frac{d}{dt_4}$. Equations (2.3.4), (2.3.5) suggest splitting the analysis of this system into the following cases:

- i) $g_2 = 0$ and $g_3 = 0$,
- ii) $g_2 \neq 0$ and $g_3 = 0$,
- iii) $g_2 = 0$ and $g_3 \neq 0$,
- iv) $g_2 \neq 0$ and $g_3 \neq 0$.

Case i) $g_2 = 0, g_3 = 0$. Then equation (2.3.6) becomes

$$g_4g_1' = g_1g_4' \Rightarrow g_4 = \mu g_1,$$

where μ is a constant of integration. Then (2.3.7) gives the evolution of γ and g_1 :

$$g_1'' = 3\gamma g_1' - \frac{3}{2}g_1\gamma^2 + 864\mu g_1^3.$$

This can also be rewritten as

$$\begin{aligned} \gamma' &= \frac{1}{2}\gamma^2 - 288\mu g_1^2, \\ g_1'' &= 3\gamma g_1' - 3g_1\gamma'. \end{aligned} \quad (2.3.8)$$

On eliminating g_1 one obtains a third-order scalar equation

$$\ddot{\gamma} = \frac{1}{2} \frac{(\dot{\gamma} - 2\gamma\dot{\gamma})^2}{\dot{\gamma} - \gamma^2} + 8\gamma\ddot{\gamma} - 10\dot{\gamma}^2. \quad (2.3.9)$$

Here the independent variable has been rescaled, $t = \frac{1}{2}t_4$, and hence $\gamma' = \frac{1}{2}\dot{\gamma}$ etc.. This falls within Bureau's class (see Section 2.4) and its solutions are given in terms of the Schwarzian triangle function $y(t) = S[\frac{1}{2}, \frac{1}{4}, 0, t]$, namely

$$\begin{aligned}\gamma(t) &= \frac{1}{2} \left\{ \frac{\ddot{y}}{\dot{y}} - \left(\frac{1/2}{y} + \frac{3/4}{y-1} \right) \dot{y} \right\}, \\ &= \frac{1}{2} \frac{d}{dt} \log \left\{ \frac{\dot{y}}{y^{\frac{1}{2}}(y-1)^{\frac{3}{4}}} \right\}, \\ g_1(t) &= \frac{1}{192\mu^{\frac{1}{2}}} \frac{1}{y^{\frac{1}{2}}(y-1)^{\frac{1}{2}}} \dot{y}.\end{aligned}$$

An alternative way to solve (2.3.9) (following Satake [58] and Example 22) is to express the solutions in terms of solutions to the Halphen system. In particular, we take

$$\begin{aligned}\gamma(t) &= \frac{1}{4}(\omega_1 + 2\omega_2 + \omega_3), \\ g_1(t) &= \frac{1}{96\mu^{\frac{1}{2}}}(\omega_1 - 2\omega_2 + \omega_3).\end{aligned}$$

The required modularity properties of γ and g_1 then follow automatically from the known modularity properties of the solution to the Halphen system.

Recall Lemma 1.4.18, that the canonical coordinates of a Frobenius manifold are the roots of the equation

$$p(u) := \det \left(g^{\alpha\beta}(t) - u\eta^{\alpha\beta} \right) = 0. \quad (2.3.10)$$

This provides a way to check whether the Frobenius manifold in question is semi-simple or not: we compute the roots of this polynomial. If the roots are all distinct the canonical coordinate system is well defined and the manifold is semi-simple. If the polynomial has repeated roots then the manifold is nilpotent. Of course, the expressions for the canonical coordinates in terms of the flat coordinates can be extremely complicated, and already in dimension 4 requires one to solve a quartic. Alternatively, one can check by computing the resultant

$$\text{res}(p, p') = \prod_{(x,y) \in \mathcal{S}} (x-y),$$

where $\mathcal{S} = \{(x,y) : p(x) = 0, p'(y) = 0\}$. If the resultant is zero, p has a repeated root and the manifold is nilpotent. If the resultant is non-zero the manifold is semi-simple. This procedure only depends on the form of the dynamical system, *not* the particular solution.

In particular, for the case in hand,

$$\text{res}(p, p') \neq 0,$$

and the solution is semi-simple.

Case ii) $g_2 \neq 0, g_3 = 0$. In this case equation (2.3.5) becomes

$$24g_1g_2g_4 = 0 \Rightarrow g_1 = 0, \text{ or } g_4 = 0 \quad (\text{as } g_2 \neq 0).$$

But if either $g_1 = 0$, or $g_4 = 0$, the evolution of γ is given by (2.3.3)

$$D\gamma = 0,$$

and any resulting solution of the WDVV equation will not give a modular Frobenius manifold, as above.

Case iii) $g_2 = 0, g_3 \neq 0$. In this case equation (2.3.4) becomes

$$24g_1g_3g_4 = 0 \Rightarrow g_1 = 0, \text{ or } g_4 = 0,$$

and we are in the same scenario as in Case i).

Case iv) $g_2 \neq 0, g_3 \neq 0$. Because g_2 and g_3 are non-zero we may divide equations (2.3.4), (2.3.5) by g_2 and g_3 respectively to obtain

$$Dg_4 = \frac{12g_1g_3g_4}{g_2} = \frac{g_4}{g_2}Dg_2; \quad Dg_1 = \frac{24g_1g_2g_4}{g_3} = \frac{g_1}{g_3}Dg_3.$$

Consider the latter of these equalities. Un-packing we obtain the relationship

$$\frac{1}{g_1}(g'_1 - \gamma g_1) = \frac{1}{g_3}(g'_3 - \gamma g_3) \Rightarrow g_1 = \mu g_3.$$

Similarly, the former gives $g_2 = \mu_2 g_4$. Analyzing the obstructions to associativity with these algebraic relationships in place we find the constraint $\mu_2 = 4/\mu$. The WDVV equations are then equivalent to modular dynamical system

$$\begin{aligned} \gamma' &= \frac{1}{2}\gamma^2 - 288g_1g_4 & D\gamma &= -288g_1g_4, \\ g'_1 &= \gamma g_1 + 24\mu g_4^2 & \Leftrightarrow Dg_1 &= 24\mu g_4^2, \\ g'_4 &= \gamma g_4 + 24\mu^{-1}g_1^2 & Dg_4 &= 24\mu^{-1}g_1^2. \end{aligned} \tag{2.3.11}$$

Eliminating g_1 and g_4 yields the Chazy equation (2.3.2). This system was originally discovered by Guerts, Martini & Post [31].

From the expressions for γ' and γ'' one can easily obtain two algebraic relations connecting g_1 and g_4 to the γ . Hence one can obtain, by solving these algebraic equations, the general solution in this subcase. Since it is well known that the Halphen system is equivalent to the Chazy equation, one may also express the solution in terms of $\{\omega_1, \omega_2, \omega_3\}$. One can also show, without first having to solve the equations, that the solution is semi-simple.

$$\mathbf{N} = \mathbf{4}, \sigma = \frac{1}{3}.$$

We have the ansatz

$$F = \frac{1}{2}t_1^2t_4 + t_1t_2t_3 - \frac{1}{4}(t_2t_3)^2\gamma(t_4) + \{t_3^6g_4(t_4) + t_2t_3^4g_3(t_4) + t_2^3g_1(t_4)\}. \quad (2.3.12)$$

Because the analysis of the WDVV equations is analogous to the above we do not give details. The WDVV equations are equivalent to the system (there are other subcases which appear in the analysis, but these yield non-semi-simple solutions):

$$\begin{aligned} \gamma' &= \frac{1}{2}\gamma^2 - 72\mu g_1^4, \\ g_1'' &= 2\gamma g_1' - g_1\gamma' \end{aligned} \quad (2.3.13)$$

where μ is a constant and

$$g_3 = \mu g_1^3, \quad g_4 = \frac{\mu g_1}{30} \left[g_1' - \frac{1}{2}g_1\gamma \right].$$

On eliminating g_1 one obtains a third-order scalar equation

$$\ddot{\gamma} = \frac{3}{4} \frac{(\dot{\gamma} - 2\gamma\dot{\gamma})^2}{\dot{\gamma} - \gamma^2} + 6\gamma\ddot{\gamma} - 6\dot{\gamma}^2. \quad (2.3.14)$$

Here the independent variable has been rescaled, $t = \frac{1}{2}t_4$, and hence $\gamma' = \frac{1}{2}\dot{\gamma}$ etc.. This falls within Bureau's class, and its solutions are given in term of the Schwarzian triangle function $y(t) = S[\frac{1}{2}, \frac{1}{6}, 0, t]$. One can check, without first solving the differential equations, that the solution is semi-simple.

$$\begin{aligned} \gamma(t) &= \frac{1}{2} \left\{ \frac{\ddot{y}}{\dot{y}} - \left(\frac{1/2}{y} + \frac{5/6}{y-1} \right) \dot{y} \right\}, \\ &= \frac{1}{2} \frac{d}{dt} \log \left\{ \frac{\dot{y}}{y^{\frac{1}{2}}(y-1)^{\frac{5}{6}}} \right\}. \end{aligned}$$

Given this solution one may easily find the remaining functions: they all take the schematic form

$$g_i(t) = \frac{c_i}{y^{a_i}(y-1)^{b_i}} (\dot{y})^{\frac{i}{2}}, \quad i = 1, 3, 4$$

for various constants a_i, b_i, c_i .

Example 20. *[Folding Verlinde & Warner's Solution] By restricting the prepotential found by Verlinde & Warner (1.5.1) to the hyperplanes*

$$t_2 = t_3 = t_4, \quad t_5 = t_6 = t_7,$$

we obtain the ansatz (2.3.12) (up to a re-scaling). This process of restricting to hyperplanes is known as 'folding' because it was shown by Zuber [70], in the case of polynomial Frobenius manifolds, that the non-simply-laced examples (namely $B_n, F_4, G_2, H_{3,4}, I_2(n)$) may be obtained by restricting the simply laced examples (namely $A_n, D_n, E_{6,7,8}$) to certain hyperplanes, which can be understood as foldings of the corresponding Dynkin diagrams by an automorphism. In order to obtain a subalgebra one requires the condition

$$c_{ij}^k \Big|_{\Sigma} = 0, \quad \forall i, j \in I, \forall k \notin I.$$

Verlinde & Warner [66] found that the WDVV equations reduce, quite remarkably, to a third order modular invariant differential equation for their pivot term

$$\gamma_W(t) = \frac{1}{2} \left\{ \frac{\beta''}{\beta'} + \left(\frac{1}{3(1-\beta)} - \frac{2}{3\beta} \right) \beta' \right\},$$

where $\beta = S[\frac{1}{3}, 0, 0; t]$. It does not appear at first sight that this solution to WDVV obtained by folding the solution (1.5.1), and the solution (2.3.14) agree - the triangle functions look completely different (recall that the function $\gamma(t)$ appearing in the solution (2.3.14) was given in terms of the triangle function $y(t) = S[\frac{1}{2}, \frac{1}{6}, 0; t]$). However, due to non-linear identities between hypergeometric functions due to Goursat [30], [43] we obtain via equation (2.4.10) identities connecting different Schwarzian triangle functions. For example, if we define

$$\begin{aligned} \beta(t) &= S[\frac{1}{3}, 0, 0; t], \\ y(t) &= S[\frac{1}{2}, \frac{1}{6}, 0; t], \end{aligned}$$

then the identity (Goursat (44) [30])

$$(1-z)^{-\frac{1}{6}} {}_2F_1 \left(\frac{1}{6}, \frac{1}{6}, \frac{5}{6}; z := \frac{x^2}{4(x-1)} \right) = {}_2F_1 \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; x \right) \Rightarrow y = \frac{\beta^2}{4(\beta-1)}.$$

σ	Non-pivot solutions $(\alpha_2, \alpha_3, \alpha_4)$:
$\frac{1}{2}$	$(4, 0, 0), (3, 0, 1), (1, 0, 3), (0, 0, 4)$ $(3, 1, 0), (2, 1, 1), (1, 1, 2), (0, 1, 3)$ $(2, 2, 0), (0, 2, 2)$ $(1, 3, 0), (0, 3, 1);$
$\frac{1}{3}$	$(3, 0, 0), (1, 0, 4), (0, 0, 6), (0, 2, 3);$
$\frac{1}{4}$	$(1, 0, 5), (0, 0, 8)$ $(2, 1, 0), (1, 1, 3), (0, 1, 6)$ $(0, 2, 4), (0, 3, 2);$
$\frac{1}{n}, (n \geq 5, \text{ and odd})$	$(1, 0, n+1), (0, 0, 2n), (0, 2, n);$
$\frac{2}{2n+1}, (n \geq 2)$	$(0, 0, 2n+1);$
$\frac{1}{2n}, (n > 2)$	$(1, 0, 2n+1), (0, 0, 4n), (1, 1, n+1), (0, 1, 3n)$ $(0, 2, 2n), (0, 3, n)$
$\frac{3}{2n}, (n \geq 4)$	$(0, 1, n)$
arbitrary	none .

Table 2.2: Non-pivot solutions of the Frobenius equation (2.2.4) for $N = 5$.

2.3.3 Modular Dynamical Systems for $N = 5$.

There is again a 1-parameter family of Euler vector fields,

$$E = t^1 \frac{\partial}{\partial t^1} + (1 - \sigma) t^2 \frac{\partial}{\partial t^2} + \frac{1}{2} t^3 \frac{\partial}{\partial t^3} + \sigma t^4 \frac{\partial}{\partial t^4}.$$

In this case we have 3 pivot points which correspond to the monomials $\{(t_2 t_4)^2, t_2 (t_3)^2 t_4, (t_3)^4\}$. Again, without loss of generality we may assume decreasing degrees and hence $\sigma \leq \frac{1}{2}$. In this case, solutions of the Frobenius equation correspond to the intersection of a plane and \mathbb{N}^3 . As σ varies this plane pivots about a line which intersects the three pivot points. Analyzing the Frobenius equation (2.2.4), we find the non-pivot terms summarized in Table 2.2.

Perhaps rather remarkably, in all cases except $\sigma = 1/2$, there exist no semi-simple modular Frobenius manifolds. The obstructions to associativity typically reduce to differential equations of the following types:

1. second order Euler-type differential equations that may be integrated explicitly to

yield a non-semi-simple solution to WDVV.

2. first order equations of the form

$$\gamma' - \frac{1}{2}\gamma^2 = 0,$$

where γ is a pivot term. As in the analysis of $N = 4$, we may reduce any such function to zero, and hence the modularity properties of any resulting Frobenius manifold will be destroyed.

$N = 5, \sigma = \frac{1}{2}$.

In this case we have the ansatz

$$\begin{aligned} F &= \frac{1}{2}t_1^2t_5 + t_1t_2t_4 + \frac{1}{2}t_2t_3^2 - \frac{1}{4}(\gamma_1(t_5)(t_2t_4)^2 + \gamma_2(t_5)t_2t_3^2t_4 + \gamma_3(t_5)t_4^4) \\ &\quad + g_1(t_5)t_2^4 + g_2(t_5)t_2^3t_4 + g_3(t_5)t_2t_4^3 + g_4(t_5)t_4^4 + g_5(t_5)t_2^3t_3 + g_6(t_5)t_2^2t_3t_4 + g_7(t_5)t_2t_3t_4^2 \\ &\quad + g_8(t_5)t_3t_4^3 + g_9(t_5)t_2^2t_3^2 + g_{10}(t_5)t_3^2t_4^2 + g_{11}(t_5)t_2t_3^3 + g_{12}(t_5)t_3^3t_4; \\ E &= t_1\frac{\partial}{\partial t_1} + \frac{1}{2}t_2\frac{\partial}{\partial t_2} + \frac{1}{2}t_3\frac{\partial}{\partial t_3} + \frac{1}{2}t_4\frac{\partial}{\partial t_4}. \end{aligned} \quad (2.3.15)$$

Though it was not possible to give a classification of all the solutions of this form, we give examples of two that have appeared in the literature.

Example 21. *The following example (rescaled slightly) was found Guerts, Martini & Post [31].*

$$\begin{aligned} F &= \frac{1}{2}t_1^2t_5 + t_1\left(t_2t_4 + \frac{1}{2}t_3^2\right) - \frac{1}{4}\left(t_2t_4 + \frac{1}{2}t_3^2\right)^2\gamma(t_5) \\ &\quad + \frac{1}{24}(t_2^4 - 2t_2t_3^3 - 2t_2t_4^3 + 3t_3^2t_4^2)h_1(t_5) + \frac{1}{24}(t_4^4 - 2t_4t_3^3 - 2t_2^3t_4 + 3t_2^2t_3^2)h_2(t_5) \end{aligned}$$

where

$$\begin{aligned} \gamma' &= \frac{1}{2}\gamma^2 - \frac{1}{2}h_1h_2, \\ h_1' &= \gamma h_1 + h_2^2, \\ h_2' &= \gamma h_2 + h_1^2. \end{aligned} \quad (2.3.16)$$

This system reduces to the Chazy equation for the function γ .

Example 22. *In the same way that we folded Verlinde & Warner's solution in Example 20, the following solution may be obtained by restricting the $D_4^{(1,1)}$ solution found by Satake [58] to certain hyperplanes $\Sigma = \{t^i = 0, i \notin I\}$ for some subset $I \subset \{1, \dots, N\}$.*

For the case in hand, we can reduce Satake's six dimensional solution to a five dimensional one by setting any one $t^i = 0$, for $i \in \{2, \dots, N-1\}$. Following this procedure we find

$$\begin{aligned} F = & \frac{1}{2}t_1^2t_5 + \frac{1}{2}t_1(t_2^2 + t_3^2 + t_4^2) - \frac{1}{16}(t_2^2 + t_3^2 + t_4^2)^2\gamma(t_5) \\ & + (t_2^4 + t_3^4 - 2t_3^2t_4^2 - 2t_2^2t_3^2 - 2t_2^2t_4^2)g_1(t_5) + \frac{1}{2}(t_2t_3t_4^2)g_2(t_5), \end{aligned}$$

where

$$\begin{aligned} \gamma' &= \frac{1}{2}\gamma^2 - \frac{2}{3}g_2^2 - 128g_1^2, \\ g_1' &= \gamma g_1 - \frac{1}{12}g_2^2 + 16g_1^2, \\ g_2' &= \gamma g_2 - 32g_1g_2. \end{aligned} \tag{2.3.17}$$

This system also reduces to the Chazy equation for the function γ . Note the slightly different form of metric in this example.

Though not obvious in their current form, it turns out that the two systems in these examples are inequivalent. Both fall into a class of differential equations called (for obvious reasons) quadratic differential equations, first studied by Lawrence [39], and more recently by Ohyama [49]. As we will see in the remainder of this chapter, one may associate to such a system an algebra in a very natural way. The systems are then analysed by studying the properties of these *associated algebras*. For the two examples presented here the corresponding algebras - to be defined in the next section - are non-isomorphic, and hence so are the modular Frobenius manifolds.

2.4 Solving Modular Dynamical Systems

We now turn our attention to the methods employed to solve the dynamical systems derived above. Upon eliminating the non-pivot functions, all the dynamical systems we have found yield a third order modular invariant differential equation for the pivot function γ . They all fall into a class of differential equations of the form

$$\ddot{u} = \alpha \frac{(\ddot{u} - 2u\dot{u})^2}{\dot{u} - u^2} + \beta u\ddot{u} + \gamma \dot{u}^2 + \delta (\dot{u} - u^2)^2, \tag{2.4.1}$$

the general solution of which was first constructed by Bureau [7]. As hinted at earlier, equations of the form (2.4.1) will be referred to as Bureau's class. The constants α, β, γ

and δ are subject to the constraints

$$16\alpha - \gamma = 18, \quad 2\beta + \gamma = 6.$$

The importance of the class of equations (2.4.1) becomes more apparent when written in terms of the Rankin derivative,

$$(Du)(D^3u) = \alpha(D^2u)^2 + (\delta + 16\alpha - 24)(Du)^3.$$

The constants are subject to the same constraints. In fact, any modular invariant differential equation of rank three and modular weight 8 can be obtained from this one by an appropriate choice of the constants. This follows from the following

Proposition 10. [16] *Let P be a polynomial. Suppose that the differential equation*

$$P(\gamma, \gamma', \gamma'', \dots, \gamma^{(k)}) = 0 \tag{2.4.2}$$

is invariant with respect to Möbius transformations of the independent variable τ . Then the differential equation (2.4.2) can be represented as

$$Q(D\gamma, D^2\gamma, \dots, D^l\gamma) = 0, \quad k = l + 2,$$

where $Q(D\gamma, D^2\gamma, \dots, D^l\gamma)$ is a homogenous polynomial in γ and its Rankin derivatives.

Proof. One direction is easy: by construction, any such Q is modular invariant. To show that such a Q exists, it is enough to consider the case where P is linear in the highest order term $\gamma^{(k)}$. Because P is modular invariant, each term must be of the same weight. Therefore it can be factorised as follows:

$$P = \prod_{i=1}^k \left(\frac{d}{d\tau} - \alpha_j \gamma \right)^i \gamma = 0.$$

Then we must show that $\alpha_j = 1/2$, for $j = 1, \dots, n$. To this end, suppose $\exists \alpha_j \neq 1/2$. Then

$$\begin{aligned} P &= \prod_{i \neq j} \left(\frac{d}{d\tau} - \frac{1}{2} \gamma \right)^i \left(\frac{d}{d\tau} - \alpha_j \gamma \right) \gamma \\ &= D^{k-1} \left(D\gamma + \left(\frac{1}{2} - \alpha_j \right) \gamma^2 \right) \\ &= D^k \gamma + 2\gamma \left(\frac{1}{2} - \alpha_j \right) D^{k-1} \gamma. \end{aligned}$$

Note that the second term is not modular invariant. So $\alpha_i = 1/2$, for $i = 1, \dots, n$. □

For example, when $\alpha = 0$, we obtain the so-called Chazy class XII,

$$D^3 u = (\delta - 24)(Du)^2,$$

with Chazy itself appearing for $\delta = 0$.

Bureau found that the solution to (2.4.1) is given in terms of a function y ,

$$\begin{aligned} u &= \frac{1}{2} \left(\frac{\ddot{y}}{\dot{y}} - q(y) \dot{y} \right) \\ &= \frac{1}{2} \frac{d}{dt} \log \left(\frac{\dot{y}}{y^{(1-n)}(y-1)^{(1-m)}} \right), \end{aligned}$$

where

$$q(y) = \left(\frac{(1-n)}{y} + \frac{(1-m)}{y-1} \right)$$

and y satisfies the *Schwarzian* differential equation

$$\frac{\ddot{y}}{\dot{y}} - \frac{3}{2} \left\{ \frac{\ddot{y}}{\dot{y}} \right\}^2 = -\frac{1}{2} \left\{ \frac{1-n^2}{y^2} + \frac{1-m^2}{(1-y)^2} - \frac{(1+p^2-m^2-n^2)}{y(y-1)} \right\} \dot{y}^2. \quad (2.4.3)$$

The relationship between $\alpha, \beta, \gamma, \delta$ and m, n, p are given by the requirement that $L_i = 0, i = 1, 2, 3$ where

$$\begin{aligned} L_1 &= (1-2n)[1-3n-\alpha(1-2n)], \\ L_2 &= (1-2m)[1-3m-\alpha(1-2m)], \\ L_3 &= (1-2m)(2-3n) + (1-2n)(2-3m) - 2\alpha(1-2m)(1-2n) \\ &\quad + \frac{1}{4}(\gamma + \delta - 6)[(1-m-n)^2 - p^2]. \end{aligned}$$

We recall briefly here how to construct solutions of the Schwarzian differential equation 2.4.3.

Consider the second order Fuchsian differential equation

$$x''(z) + I(z)x(z) = 0. \quad (2.4.4)$$

Fuchsian means that the function $I(z)$ has poles of order at most 2. One may keep in mind the hypergeometric equation, as three regular singular points will suffice for our needs. Let x_1 and x_2 be any two linearly independent solutions to (2.4.4). Then

$$x_1''x_2 - x_1x_2'' = 0 \Rightarrow x_1'x_2 - x_1x_2' = c, \quad c \neq 0 \text{ constant.}$$

That is, the Wronskian of the solutions is constant. This constant is non-zero because the

solutions are linearly independent. Now consider their ratio,

$$\tau(z) = \frac{x_2(z)}{x_1(z)}.$$

For example, in the case of the hypergeometric equation,

$$z(z-1)x'' + [(a+b+1)z-c]x' + abx = 0, \quad (2.4.5)$$

we have

$$\tau(z) = \frac{z^{1-c} {}_2F_1(1+a-c, 1+b-c, 2-c, z)}{{}_2F_1(a, b, c, z)}.$$

The inverses of ratios of hypergeometric functions like this are known as *Schwarzian triangle functions*. Due to the constancy of the Wronskian, it follows that τ satisfies the Schwarzian differential equation,

$$\frac{\tau'''}{\tau'} - \frac{3}{2} \left(\frac{\tau''}{\tau'} \right)^2 = 2I(z). \quad (2.4.6)$$

To introduce some terminology, the left hand side of (2.4.6) is known as the *Schwarzian derivative* of τ with respect to z . It is denoted

$$\{\tau, z\} = \frac{\tau'''}{\tau'} - \frac{3}{2} \left(\frac{\tau''}{\tau'} \right)^2.$$

Two key properties are

- The Schwarzian is a projective invariant:

$$\left\{ \frac{a\tau + b}{c\tau + d}, z \right\} = \{\tau, z\},$$

for $ad - bc \neq 0$.

- The Schwarzian satisfies the *connection formula*: If $w(z)$ is a non-constant function of z ,

$$\{\tau, w\} = \{z, w\} + \{\tau, z\} \left(\frac{dz}{dw} \right)^2.$$

An equivalent interpretation of the former property is

$$\{\tau, x\} = 0 \Leftrightarrow x = \frac{a\tau + b}{c\tau + d}.$$

For the case in hand, the Schwarzian trivializes the monodromy of the ODE (2.4.4): If one analytically continues a pair of solutions around a loop encircling a singular point of

the differential equation, we obtain two new solutions related to the original ones by an invertible linear transformation,

$$\left. \begin{aligned} x_1(z) &\mapsto ax_1(z) + bx_2(z) \\ x_2(z) &\mapsto cx_1(z) + dx_2(z) \end{aligned} \right\} \Rightarrow \tau(z) \mapsto \frac{a\tau(z) + b}{c\tau(z) + d}.$$

From the latter property it follows that

$$\{z, \tau\} = - \left(\frac{dz}{d\tau} \right)^2 \{\tau, z\}$$

if we take $\tau = w$, since $\{\tau, \tau\} = 0$. Although both properties have a deep geometric meaning, they can be checked directly by calculation.

Using the second property, we find that the inverse function $z(\tau)$ satisfies the *nonlinear* differential equation

$$\{z, \tau\} = -I(z) \left(\frac{dz}{d\tau} \right)^2.$$

This is exactly the equation (2.4.3), for the choice

$$I(z) = -\frac{1}{2} \left\{ \frac{1-n^2}{z^2} + \frac{1-m^2}{(1-z)^2} - \frac{(1+p^2-m^2-n^2)}{z(z-1)} \right\}.$$

So starting from an appropriate second order Fuchsian differential equation, we may construct, by taking the inverse of the ratio of two solutions, solutions to the Schwarzian differential equation (2.4.3).

The constraints $L_i = 0$, for $i = 1, 2, 3$ on the various parameters, although not immediately obvious, follow from

Proposition 11 (Due to I.A.B. Strachan (2011), unpublished). *Suppose*

$$u = \frac{1}{2} \left\{ \frac{\ddot{z}}{\dot{z}} - q(z)\dot{z} \right\}$$

where z satisfies the Schwarzian differential equation

$$\{z, \tau\} = -\frac{1}{2}I(z)\dot{z}^2. \quad (2.4.7)$$

Then

$$D^n u = h_n(z)\dot{z}^{n+1}, \quad n \in \mathbb{N},$$

for

$$h_n(z) = \dot{h}_{n-1}(z) + nq(z)h_{n-1}(z).$$

Proof. Proof is carried out by induction. We have

$$\begin{aligned}
 Du = \dot{u} - u^2 &= \frac{1}{2} \left(\frac{\ddot{z}}{\dot{z}} - \left(\frac{\ddot{z}}{\dot{z}} \right)^2 - q'(z) - 2q(z)\ddot{z} \right) - \frac{1}{4} \left(\frac{\ddot{z}}{\dot{z}} - q(z)\dot{z} \right)^2, \\
 &= \frac{1}{2} \left(\frac{\ddot{z}}{\dot{z}} - \frac{3}{2} \left(\frac{\ddot{z}}{\dot{z}} \right)^2 \right) - \left(\frac{1}{2}q'(z) + \frac{1}{4}q(z)^2 \right) \dot{z}^2, \\
 &= \left(-\frac{1}{4}I(z) - \frac{1}{2}q'(z) + \frac{1}{4}q(z)^2 \right) \dot{z}^2 = h_1(z)\dot{z}^2.
 \end{aligned}$$

So the assertion holds for $n = 1$. Suppose now that $D^n u = h_n(z)\dot{z}^{n+1}$. Then, by definition of the Rankin derivative, we have $D^n u = D(D^{n-1}u) - 2nuD^{n-1}u$, and so

$$D^{n+1}u = h'_n(z)\dot{z}^{n+2} + h_n(z)(n+1)\dot{z}^n\ddot{z} - 2u(n+1)h_n(z)\dot{z}^{n+1}.$$

By definition, $\ddot{z} = 2u\dot{z} + q(z)\dot{z}^2$, and so

$$\begin{aligned}
 D^{n+1}u &= h'_n(z)\dot{z}^{n+2} + (n+1)h_n(z)\dot{z}^n(2u\dot{z} + q(z)\dot{z}^2) - 2u(n+1)h_n(z)\dot{z}^{n+1} \\
 &= (h'_n(z) + (n+1)h_n(z)q(z))\dot{z}^{n+2} \\
 &= h_{n+1}(z)\dot{z}^{n+2}.
 \end{aligned}$$

□

This proposition turns modular invariant differential equations into algebraic equations, and gives a simplification of Bureau's original argument. For example, Bureau's equation (2.4.1) becomes

$$h_1(z)h_3(z) = \alpha h_2(z)^2 + (\delta + 16\alpha - 24)h_1(z)^3. \quad (2.4.8)$$

For $I(z)$ and $q(z)$ defined above one can easily check that

$$h_1(z) = -\frac{1}{4} \left(\frac{(1-m-n)^2 - p^2}{z(z-1)} \right) =: -\frac{1}{4} \frac{M}{z(z-1)}, \quad (2.4.9)$$

for $M = (1-m-n)^2 - p^2$. Using the recursion relation derived above, one can compute

$$\begin{aligned}
 h_2(z) &= -\frac{1}{4}M \left(\frac{1-2n}{z^2(z-1)} + \frac{1-2m}{z(z-1)^2} \right), \\
 h_3(z) &= -\frac{1}{4}M \left(\frac{1}{z^3(z-1)} (3(1-n)(1-2n) - 2(1-2n)) \right. \\
 &\quad \left. + \frac{1}{z(z-1)^3} (3(1-m)(1-2m) - 2(1-2m)) \right. \\
 &\quad \left. + \frac{1}{z^2(z-1)^2} ((2n+2m-2) + 3(1-n)(1-2m) + 3(1-m)(1-2n)) \right).
 \end{aligned}$$

Substituting into (2.4.8) and equating powers of z , we arrive at Bureau's constraints $L_i = 0$, for $i = 1, 2, 3$.

Let us summarize our strategy for solving the dynamical systems derived above. Note that although a lot of the theory set out above applies to modular differential equations of arbitrary order, all of the dynamical systems we have found are of rank three.

Step 1. Solve the constraints $L_i = 0$, for $i = 1, 2, 3$ to identify the parameters m, n , and p to appear in the triangle function $\tau(z) = S[m, n, p, z]$ that satisfies the differential equation

$$\{\tau, z\} = I(z),$$

in terms of the coefficients appearing in the third order modular differential equation (2.4.1).

Step 2. Construct the solution $S[m, n, p, z]$ implicitly as the ratio of two linearly independent solutions of the Hypergeometric equation

$$\tau(z) = S^{-1}[m, m, p, z] = z^{1-c} \frac{{}_2F_1(1+a-c, 1+b-c, 2-c, z)}{{}_2F_1(a, b, c, z)} \quad (2.4.10)$$

The parameters m, n and p appearing in the triangle function found by solving $L_i = 0$, for $i = 1, 2, 3$ then determine the parameters of the hypergeometric function according to [7]

$$n^2 = (1-c)^2, \quad m^2 = (a+b-c)^2, \quad p^2 = (a-b)^2. \quad (2.4.11)$$

This is done by writing the hypergeometric equation in Q -form.

Step 3. Using the properties of the Schwarzian derivative, we know that the inverse function $z(\tau) = S[m, n, p, \tau]$ satisfies the *nonlinear* differential equation

$$\{z, \tau\} = -I(z)\dot{z}^2,$$

and so define

$$u = \frac{1}{2} \left\{ \frac{\ddot{z}}{\dot{z}} - q(z)\dot{z} \right\},$$

which satisfies the third order equation modular invariant differential equation (2.4.1).

Step 4. Use the dynamical system to find expressions for the non-pivot terms in terms of the solution to (2.4.1). The technicalities of this step are simplified thanks to the Proposition 11.

Let us illustrate the above procedure for the 4 dimensional Frobenius manifold with $\sigma = 1/3$. Recall that the pivot γ satisfies the third order equation (2.3.14)

$$\ddot{\gamma} = \frac{3}{4} \frac{(\ddot{\gamma} - 2\gamma\dot{\gamma})^2}{\dot{\gamma} - \gamma^2} + 6\gamma\ddot{\gamma} - 6\dot{\gamma}^2.$$

In terms of the Rankin derivative,

$$D\gamma(D^3\gamma) = \frac{3}{4}(D^2\gamma)^2 - 12(D\gamma)^3.$$

This is equivalent to the algebraic constraints

$$\begin{aligned} (1-2n)(1-6n) &= 0, \\ (1-2m)(1-6m) &= 0, \\ (2-3m)m + (2-3n)n + 3p^2 - \frac{1}{2} &= 0 \end{aligned}$$

on m, n and p . Solving the system, we find $p = 0$, and we can choose $(m, n) = (\frac{1}{2}, \frac{1}{6})$. The function q is given by

$$q(z) = \frac{1}{2z} + \frac{5}{6(z-1)}.$$

The solution to 3rd order equation (2.3.14) is given in terms of the Schwarzian triangle function[†] $z(\tau) = S[\frac{1}{2}, \frac{1}{6}, 0, \tau]$:

$$\gamma(\tau) = \frac{1}{2} \frac{d}{d\tau} \left(\frac{\dot{z}}{z^{\frac{1}{2}}(1-z)^{\frac{5}{6}}} \right).$$

We recover the non-pivot terms from the dynamical system with the aid of Proposition 11. Recall that the system was

$$\begin{aligned} \gamma' &= \frac{1}{2}\gamma^2 - 72\mu g_1^4, \\ g_1'' &= 2\gamma g_1' - g_1\gamma' \end{aligned} \tag{2.4.12}$$

where μ is a constant and

$$g_3 = \mu g_1^3, \quad g_4 = \frac{\mu g_1}{30} \left[g_1' - \frac{1}{2} g_1 \gamma \right]. \tag{2.4.13}$$

Taking into account the re-scaling of the independent variable $t = \frac{1}{2}t_4$, the differential

[†]Note that we have freedom of choice here between the values of m and n . $(m, n) = (\frac{1}{2}, \frac{1}{2})$ corresponds to the trivial solution $u = 0$. Aside from this choice, the other values give equivalent solutions due to well known identities between hypergeometric functions due to Euler, Kummer and Goursat.

equation for γ becomes

$$\begin{aligned}
 D\gamma &= -144\mu g_1^4 \\
 &= h_1(z)\dot{z}^2 \quad (\text{due to Proposition 11}) \\
 &= -\frac{1}{4}\frac{M}{z(z-1)}\dot{z}^2 \quad (\text{using (2.4.9)}) \\
 &= -\frac{1}{36}\frac{\dot{z}^2}{z(z-1)} \\
 \Rightarrow g_1 &= \frac{1}{5184^{\frac{1}{4}}\mu^{\frac{1}{4}}}\left(\frac{\dot{z}^{\frac{1}{2}}}{z^{\frac{1}{4}}(z-1)^{\frac{1}{4}}}\right).
 \end{aligned}$$

Using this, the other equations (2.4.13) imply

$$g_3 = \frac{\mu^{\frac{3}{4}}}{5184^{\frac{3}{4}}}\left(\frac{\dot{z}^{\frac{3}{2}}}{z^{\frac{3}{4}}(z-1)^{\frac{3}{4}}}\right), \quad g_4 = \frac{\mu^{\frac{1}{2}}}{25920}\left(\frac{\dot{z}^2}{z^{\frac{3}{2}}(z-1)^{\frac{1}{2}}}\right).$$

Note that as pointed out above, the non-pivot functions take the schematic form

$$g_i(\tau) = \frac{c_i}{z^{a_i}(z-1)^{b_i}}\dot{z}^{\frac{i}{2}}.$$

So we are able to solve the modular dynamical systems in terms of Schwarzian triangle functions. Let us now turn our attention to the rather elegant approach, initially due to Lawrence [39] and then extended by Ohyaama [49], for studying *quadratic systems*.

2.5 Modular Dynamical Systems and Non-Associative Algebras

Motivated by the form of the dynamical systems found above we make

Definition 12. [39] *A quadratic system of differential equations takes the form*

$$\frac{dx^i}{dt} = \sum_{j,k=1}^n a_{jk}^i x^j x^k, \quad i = 1, \dots, n, \quad (2.5.1)$$

where the n^3 constants a_{jk}^i satisfy $a_{jk}^i = a_{kj}^i$.

To understand the relevance of such systems, one should keep in mind the set of functions $\{\gamma, g_1, \dots, g_k\}$. For example, the systems (2.3.16), (2.3.17), and (2.3.11) are of this form. Lawrence's idea was to study the following algebra:

Definition 13. [39] *The associated linear algebra is defined by*

$$X_i \circ X_j = \sum_{k=1}^n a_{ij}^k X_k. \quad (2.5.2)$$

The algebra is called linear because any element can be written as a linear combination of the basis elements $\{X_1, \dots, X_n\}$. Note that the algebra is commutative due to the relation $a_{jk}^i = a_{kj}^i$, but it is not necessarily associative. Demanding associativity of the algebra places quadratic constraints on the structure constants a_{jk}^i , just like the WDVV equations. As noted above, for polynomial modular Frobenius manifolds, the WDVV equations are equivalent to a modular dynamical system. For the case when this is a quadratic system, it is an interesting problem to find an explicit formula for the constants a_{jk}^i in terms of those of the Frobenius algebras.

Example 23. *Consider the system (2.3.16) of Example 21:*

$$\begin{aligned} \gamma' &= \frac{1}{2}\gamma^2 - \frac{1}{2}h_1h_2, \\ h_1' &= \gamma h_1 + h_2^2, \\ h_2' &= \gamma h_2 + h_1^2. \end{aligned}$$

The associated algebra is given by the multiplication table:

\circ	Γ	H_1	H_2
Γ	Γ	H_1	H_2
H_1	H_1	$2H_2$	$-\frac{1}{2}\Gamma$
H_2	H_2	$-\frac{1}{2}\Gamma$	$2H_1$

(2.5.3)

So the algebra is unital, with unity Γ .

In fact, it follows from the definition of the Rankin derivative that quadratic systems that are also modular invariant are unital. For a polynomial modular Frobenius manifold, this is proportional to the pivot function: Suppose we have a set $\{\gamma, g_1, \dots, g_n\}$ of $n+1$ functions in the variable τ . Define an $SL(2, \mathbb{C})$ action by

$$\begin{aligned} \gamma\left(\frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^2\gamma(\tau) + 2c(c\tau+d), \\ g_i\left(\frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^2g_i(\tau), \quad \text{for } i = 1, \dots, n. \end{aligned}$$

Then any modular dynamical system in the variables $\{\gamma, g_1, \dots, g_n\}$ takes the form

$$D\gamma = \sum_{i,j=1}^n a_{ij}^\gamma g_i g_j,$$

$$Dg_i = \sum_{j,k=1}^n a_{jk}^i g_j g_k.$$

i.e. a quadratic system. Furthermore, the associated algebra is unital, with unity $\tilde{\Gamma} = \frac{1}{2}\Gamma$. This can be seen by a direct calculation.

As promised earlier, the study of these associated algebras would allow us to decide whether or not the two Frobenius manifolds (2.3.16), (2.3.17) are the same. The following theorem provides a large step towards answering this question:

Theorem 5. [39] *Two quadratic systems*

$$\frac{dx^i}{dt} = \sum_{j,k=1}^n a_{jk}^i x^j x^k, \quad i = 1, \dots, n, \quad (2.5.4)$$

$$\frac{dy^i}{dt} = \sum_{j,k=1}^n a_{jk}^i y^j y^k, \quad i = 1, \dots, n, \quad (2.5.5)$$

are equivalent under a nonsingular linear change of variables

$$x_i = \sum_{j=1}^n b_i^j y_j$$

if and only if their respective associated algebras are isomorphic.

Therefore we need to determine whether or not the algebras associated to the two solutions in question are isomorphic. The calculation will be made easier by choosing a basis of nilpotent elements for the algebras. It will be useful to have at hand for reference the algebra associated to the solution found by folding that of Satake (2.3.17),

\circ	Γ	G_1	G_2	(2.5.6)
Γ	Γ	G_1	G_2	
G_1	G_1	$-4\Gamma - 4G_1$	G_2	
G_2	G_2	$4G_2$	$-4\Gamma - 4G_1$	

Recall that nilpotent elements of an algebra satisfy $H^2 = 0$. Solving this equation for the algebras (2.5.3), (2.5.6), and re-writing the multiplication tables in these bases ($\{E, F, H\}$

and $\{\tilde{E}, \tilde{F}, \tilde{H}\}$ respectively), we find

$$(2.5.3) \mapsto \begin{array}{c|ccc} \circ & E & F & H \\ \hline E & E & F & H \\ F & F & 0 & \frac{1}{2}E + F + H \\ H & H & \frac{1}{2}E + F + H & 0 \end{array} \quad (2.5.6) \mapsto \begin{array}{c|ccc} \circ & \tilde{E} & \tilde{F} & \tilde{H} \\ \hline \tilde{E} & \tilde{E} & \tilde{F} & \tilde{H} \\ \tilde{F} & \tilde{F} & 0 & -\frac{1}{2}\tilde{E} + \tilde{F} + \tilde{H} \\ \tilde{H} & \tilde{H} & -\frac{1}{2}\tilde{E} + \tilde{F} + \tilde{H} & 0 \end{array}$$

A quick calculation then shows that there does not exist an isomorphism between these algebras. Hence the functions in the respective modular dynamical systems are different, and the Frobenius manifolds are non-isomorphic. To summarize, we have proved

Proposition 12. *The polynomial modular Frobenius manifolds corresponding to the modular dynamical systems (2.3.16) and (2.3.17) are not isomorphic.*

It is interesting that all of the modular dynamical systems appearing in this thesis are of rank three. Even for the solution of Verlinde & Warner [66], which is an 8 dimensional Frobenius manifold, the WDVV equations reduce to a modular dynamical system of rank three. A priori, there is no reason why this should be the case - with the ansatz (1.5.1) there is no reason why the underlying system could not be of rank 13, for example. If further progress in the classification of modular Frobenius manifolds is to be made, this phenomenon must be understood.

Frobenius manifold structures on Hurwitz spaces are some of the best understood [16], with modular Frobenius manifolds being important in the study of so-called genus one Hurwitz spaces. These are moduli spaces of algebraic curves of genus one with a fixed meromorphic function of degree $N + 1$ that realises the curve as a $N + 1$ sheet covering of \mathbb{P}^1 . It would be interesting to try to construct such meromorphic functions for the modular Frobenius manifolds presented here.

To summarise, we have defined a class of Frobenius manifolds whose prepotential is a quasi-modular function, and set out a program for classification. We have provided perspective on how existing examples in the literature fit into the framework presented here. Complete classification results for dimensions three and four have been presented, as well as partial results for dimension five. Further we have shown how to decide whether polynomial modular Frobenius manifolds are isomorphic.

The next challenge will be to look at the principal hierarchy of a given modular Frobenius manifold. We will study how the modular symmetry present in the prepotential is inherited by this hierarchy. As we will see, this provides examples of integrable systems with a modular symmetry.

Chapter 3

From Frobenius Manifolds to Integrable Systems

3.1 Infinite Dimensional Hamiltonian Systems

At the very least, an integrable equation is a non-linear (partial or ordinary) differential equation that can be solved exactly. We have already seen an example of an integrable system in Chapter 1: the WDVV equations. The preceding chapters should have convinced the reader that a more optimistic or fruitful point of view would be to understand why an equation is integrable, and what meaning its solutions have. This Chapter explores how the Frobenius manifolds themselves can be viewed as a geometric structure on the space of dependent variables of a system evolutionary partial differential equations of hydrodynamic type (sometimes called the target space of the system). The archetypal PDE of this type is the dispersionless KdV equation, thus providing the perfect platform from which to introduce many key concepts.

3.1.1 The Korteweg-de Vries Equation

The KdV equation,

$$u_T = 6uu_X + u_{XXX} \quad (3.1.1)$$

was originally derived in 1895 by Korteweg and de Vries, after whom it is named. They were interested in modelling the propagation of waves in a shallow channel of water. Any solution $u(X, T)$ is a function of both space X and time T . Equations like (3.1.1) that involve only a first order time derivative are sometimes called *evolution equations*. One may look for solutions that travel to the right with constant speed c : $u(X, T) = f(X + cT)$. With this ansatz, one obtains the ordinary differential equation

$$(f')^2 = cf^2 - 2f^3$$

Solving subject to the boundary conditions that $f, f', f'' \rightarrow 0$ as $X + cT \rightarrow \pm\infty$, one obtains the solution

$$u(X, T) = \frac{c}{2} \operatorname{sech}^2 \left(\frac{\sqrt{c}}{2} (X + cT + \phi) \right). \quad (3.1.2)$$

This sech^2 profile models the shape of a wave moving to the right with constant speed c . Note that the speed with which such a wave moves depends on its amplitude. An interesting feature of these solutions is the way they interact. One would expect that if two waves were to collide in a shallow canal of water the result would be a plethora of discombobulated sloshings. However, it was shown by Kruskal and Zabusky that this is not the case. Amazingly, they found that two solutions of the form (3.1.2) could pass through one another unperturbed, up to a shift in phase. This led to the coining of the term '*soliton*' for solutions of this form due to their particle-like interactions.

Over the past three decades there has been fresh interest in the area of integrable

systems from mathematicians working across a wide range of topics. Of particular interest are evolutionary PDEs like the KdV equation. This is primarily due to the work of E. Witten and M. Kontsevich that establishes deep connections between the KdV hierarchy and intersection theory on moduli space of Riemann surfaces satisfying certain stability conditions. This connection has been exploited to answer very old questions arising in the field of enumerative geometry.

3.1.2 Conservation Laws of the KdV Equation

A key property of the KdV equation is the existence of conservation laws. A conservation law is an equation of the form

$$\partial_T h(u, u_X, u_{XX}, \dots) = \partial_X f(u, u_X, u_{XX}, \dots). \quad (3.1.3)$$

The reason for this choice of name becomes apparent if one considers the function

$$\mathcal{H} = \int_{\mathcal{D}} h(u, u_X, u_{XX}, \dots) dX.$$

Taking the time derivative, we have

$$\partial_T \int_{\mathcal{D}} h(u, u_X, u_{XX}, \dots) dX = \int_{\mathcal{D}} \partial_X f(u, u_X, u_{XX}, \dots) dX = f(u, u_X, u_{XX}, \dots)|_{\partial \mathcal{D}}.$$

If the boundary conditions are chosen in such a way that $f(u, u_X, u_{XX}, \dots)|_{\partial \mathcal{D}} = 0$, then we see that the functional

$$\mathcal{H} = \int_{\mathcal{D}} h(u, u_X, u_{XX}, \dots) dX$$

is a constant of evolution. Such constants of evolution are known as *Hamiltonians*. Usually, it is specified that the spatial domain of integration is \mathbb{R} , subject to $u, u_X, u_{XX}, \dots \rightarrow 0$ as $X \rightarrow \pm\infty$, as in the construction of soliton solutions (3.1.2) of the KdV equation. From a mathematical point of view, integration over a spatial domain which is a closed manifold like S^1 provides an adequate model in which to make sense of conservation laws as well.

Example 24. Note that the KdV equation itself can be written as a conservation law,

$$\partial_T (u) = \partial_X (3u^2 + u_{XX}), \quad (3.1.4)$$

so that $\mathcal{H}_1 = \int u dX$ is a conserved quantity. This conservation law has the physical interpretation of conservation of mass. We also have

$$\begin{aligned} \partial_T \left(\frac{1}{2} u^2 \right) &= uu_T = u(6uu_X + u_{XXX}) \\ &= 6u^2 u_X + uu_{XXX} = \partial_X \left(uu_{XX} + 2u^3 - \frac{1}{2} u_X^2 \right), \end{aligned} \quad (3.1.5)$$

so that $\mathcal{H}_1 = \int \frac{1}{2} u^2 dX$ is also a conserved quantity, corresponding to conservation of momentum. Finally, one can check that

$$\partial_T \left(u^3 - \frac{1}{2} u_X^2 \right) = \partial_X \left(-u_X u_{XXX} + \frac{1}{2} u_{XX}^2 + 3u^2 u_{XX} - 6uu_X^2 + \frac{9}{2} u^4 \right) \quad (3.1.6)$$

which corresponds to conservation of total energy, $\mathcal{H}_3 = \int (u^3 - \frac{1}{2} u_X^2) dX$. What makes the KdV equation special is that as well as these physically intuitive conserved quantities, it admits infinitely many conservation laws. This statement was conjectured by Miura, and later proved by Gardner. For a proof of this result see, for example, the book of Drazin and Johnson [15].

The existence of infinitely many conservation laws for the KdV equation may be interpreted in another way: the KdV equation is a member of an infinite *hierarchy* of commuting partial differential equations, generated by its conservation laws. The other ingredient involved in generating this hierarchy will be a *Poisson bracket*.

3.1.3 Poisson Brackets on Loop Spaces

Let us set out a framework with which we can extend many of the integrable characteristics we have observed for the KdV hierarchy to other evolutionary PDEs. For a more detailed exposition of this setup, see [48, 24].

Let \mathcal{M} be a smooth manifold of dimension N . Consider the space $L(\mathcal{M}) = \text{Maps}(S^1, \mathcal{M})$ of smooth maps from S^1 into \mathcal{M} . If X is our coordinate on S^1 , any element of $L(\mathcal{M})$ is represented by an N -tuple $(v^1(X), \dots, v^N(X))$. The coordinate $X \in S^1$ will play the role of the independent spatial variable in the PDEs we will construct.

Motivated by the conservation laws of the KdV equation, we are interested in functionals over our loop space,

$$\mathcal{F} = \int_{S^1} f(v, v_X, v_{XX}, \dots, v_{kX}) dX, \quad (3.1.7)$$

whose density depends on derivatives of the coordinates v with respect to X of order up to $k < \infty$. The ‘Euler-Lagrange’ or ‘variational’ derivative is defined on functionals of this form by

$$\delta \mathcal{F} = \int_{S^1} \frac{\delta \mathcal{F}}{\delta v^\sigma} \delta v^\sigma(X) dX$$

where

$$\frac{\delta \mathcal{F}}{\delta v^\sigma} = \frac{\partial f}{\partial v^\sigma} - \partial_X \left(\frac{\partial f}{\partial v_X^\sigma} \right) + \dots + (-1)^k \partial_X^k \left(\frac{\partial f}{\partial v_{kX}^\sigma} \right).$$

Definition 14. A Poisson bracket on the space of functionals on the loop space $L(\mathcal{M})$ is a bilinear pairing satisfying, for any three functionals \mathcal{H}, \mathcal{F} , and \mathcal{G} of the form (3.1.7)

1. *Anti-symmetry,*

$$\{\mathcal{F}, \mathcal{G}\} = -\{\mathcal{G}, \mathcal{F}\},$$

2. *The Jacobi identity,*

$$\{\{\mathcal{F}, \mathcal{G}\}, \mathcal{H}\} + \{\{\mathcal{H}, \mathcal{F}\}, \mathcal{G}\} + \{\{\mathcal{G}, \mathcal{H}\}, \mathcal{F}\} = 0,$$

3. *The Leibniz rule,*

$$\{\mathcal{F}, \mathcal{G}\mathcal{H}\} = \mathcal{G}\{\mathcal{F}, \mathcal{H}\} + \mathcal{H}\{\mathcal{F}, \mathcal{G}\}.$$

There is an obvious grading on the space of functionals that counts the number of X -derivatives. This can be made explicit by re-scaling the spatial variable, $X \mapsto \varepsilon X$, for $|\varepsilon|$ small. This induces a mapping on the operator ∂_X :

$$X \mapsto \varepsilon X, \quad \partial_X \mapsto \varepsilon \partial_X,$$

and so powers of ε ‘count’ X -derivatives. A Poisson bracket of order κ on the space of functionals is given by

$$\{\mathcal{F}, \mathcal{H}\} = \int_{S^1} \frac{\delta \mathcal{F}}{\delta v^\sigma} \sum_{k=0}^{k_1} A_k^{\sigma\nu}(v, v_X, \dots, v_{k_2 X}) \partial_X^k \left(\frac{\delta \mathcal{H}}{\delta v^\nu} \right) dX,$$

where $\kappa = \max\{k_1 + k_2 : k = 0, \dots, k_1\}$. The conditions that this defines a Poisson bracket then place very strict constraints on the coefficients $A_k^{\sigma\nu}(v, v_X, \dots, v_{k_2 X})$.

Example 25 ($\kappa = 0$). *Let us study Poisson brackets on our loop space of order zero,*

$$\{\mathcal{F}, \mathcal{H}\} = \int_{S^1} \frac{\delta \mathcal{F}}{\delta v^\sigma} \sum_{k=0}^{k_1} A^{\sigma\nu}(v) \frac{\delta \mathcal{H}}{\delta v^\nu} dX.$$

Such Poisson brackets are generated by Poisson brackets on $C^\infty(\mathcal{M})$. For example, if the underlying target manifold \mathcal{M} admits a symplectic form $\omega \in \wedge^2(\mathcal{M})$, the assignment

$$A^{\sigma\nu} = \omega^{\sigma\nu}$$

defines a Poisson bracket over the space of functionals. In the case where our functional densities depend only on the coordinate fields and not their derivatives,

$$\mathcal{F} = \int_{S^1} f(u) dX,$$

we generate Hamilton’s equations of motion,

$$df = \iota_v \omega.$$

What happens when $\kappa \geq 1$?

3.1.4 Poisson Brackets of Hydrodynamic Type

Let us consider Poisson brackets that are homogeneous of degree one. These take the form

$$\{\mathcal{H}, \mathcal{F}\} = \int_{S^1} \frac{\partial h}{\partial v^\alpha} \left(g^{\alpha\beta} \partial_X + b_{\kappa}^{\alpha\beta} v_{,X}^\kappa \right) \frac{\partial f}{\partial v^\beta} dX, \quad (3.1.8)$$

for any two functionals

$$\mathcal{H} = \int_{S^1} h(u) dX, \quad \mathcal{F} = \int_{S^1} f(u) dX, \quad (3.1.9)$$

whose densities are functions of the coordinates $\{v^\alpha : \alpha = 1, \dots, N\}$ and *not* their X -derivatives. We will call Poisson brackets (functionals) of this form *hydrodynamic type* Poisson brackets (functionals). This is because the Hamiltonian system they generate takes the form of a hydrodynamic type PDE:

$$\partial_T v^\alpha = \{u^\alpha(X), \mathcal{H}\} = \left(g^{\alpha\beta} \frac{\partial^2 h}{\partial v^\beta \partial v^\sigma} + b_{\sigma}^{\alpha\beta} \frac{\partial h}{\partial v^\beta} \right) \partial_X v^\sigma \quad (3.1.10)$$

Recall that in order for the bi-linear pairing (3.1.8) to define a Poisson bracket it must satisfy, in particular, the Jacobi identity:

$$\{\{\mathcal{H}, \mathcal{F}\}, \mathcal{H}\} + \{\{\mathcal{H}, \mathcal{H}\}, \mathcal{F}\} + \{\{\mathcal{F}, \mathcal{H}\}, \mathcal{H}\} = 0.$$

Dubrovin and Novikov [21] found the following differential-geometric interpretation of this constraint: if the matrix $g^{\alpha\beta}$ is non-degenerate, the bracket (3.1.8) satisfies the Jacobi identity iff

- $g^{\alpha\beta}$ define the inverse of a flat metric;
- $b_{\gamma}^{\alpha\beta} = -g^{\alpha\sigma} \Gamma_{\sigma\kappa}^{\beta}$, where $\Gamma_{\sigma\kappa}^{\beta}$ are the components of the Levi-Civita connection for the flat metric g .

Although the result is very elegant, the proof is technical and will be omitted. The reader is referred to their original paper [21], or the lectures of Hitchin for a slightly more intrinsic approach [35]. Hence by choosing a system of *flat* coordinates for the metric g , the Christoffel symbols vanish and we are left with

$$\{\mathcal{H}, \mathcal{F}\} = \int_{S^1} \frac{\partial h}{\partial v^\alpha} g^{\alpha\beta} \partial_X \left(\frac{\partial f}{\partial v^\beta} \right) dX. \quad (3.1.11)$$

In this coordinate system skew-symmetry of the bracket and the Leibniz rule are more or less immediate from integration by parts. This flat coordinate system is the set of Darboux coordinates for Poisson brackets of hydrodynamic type. For example, the first Hamiltonian structure of the KdV equation was of this type.

Note that the flat coordinates $\{v^\sigma : \sigma = 1, \dots, N\}$ of the metric g ,

$$v^\sigma(X) = \int_{S^1} v^\sigma(Y) \delta(X - Y) dY$$

span the centre of the bracket (3.1.8).

Example 26 (The KdV Hierarchy). As was noted above, the KdV equation itself may be written as a conservation law. Let us see how the KdV equation fits into an infinite hierarchy of PDEs. For this example, we take the target manifold $\mathcal{M} = \mathbb{R}$. The Gardner-Zakharov-Faddeev (GZF) bracket is defined by

$$\{\mathcal{F}, \mathcal{H}\}_{\text{GZF}} = \int_{S^1} \frac{\delta \mathcal{F}}{\delta u} \partial_X \left(\frac{\delta \mathcal{H}}{\delta u} \right) dX.$$

Then the conservation law (3.1.4) may be written in Hamiltonian form

$$\text{KdV}_1 \quad : \quad \partial_{T^1} u = \{u, \mathcal{H}_2\}_{\text{GZF}} = \partial_X \left(\frac{\delta \mathcal{H}_2}{\delta u} \right).$$

The KdV hierarchy is given by

$$\text{KdV}_n \quad : \quad \partial_{T^n} u = \{u, \mathcal{H}_{n+1}\}_{\text{GZF}}, \quad n \geq 0.$$

The flows of the hierarchy commute,

$$\partial_{T^n} \partial_{T^m} u = \partial_{T^m} \partial_{T^n} u,$$

due to the Jacobi identity and the fact that $\{\mathcal{H}_n, \mathcal{H}_m\} = 0, \forall n, m \geq 0$. The KdV hierarchy is also Hamiltonian with respect to a second Poisson bracket (though this one is not of Hydrodynamic type) called the Lenard-Magri (LM) bracket,

$$\{\mathcal{F}, \mathcal{H}\}_{\text{LM}} = \int_{S^1} \frac{\delta \mathcal{F}}{\delta u} (\partial_X^3 + 4u \partial_X + 2u_X) \left(\frac{\delta \mathcal{H}}{\delta u} \right) dX,$$

but with a shift in the indices appearing,

$$\text{KdV}_n \quad : \quad \partial_{T^n} u = \{u, \mathcal{H}_n\}_{\text{LM}}.$$

For example, the KdV equation is given by

$$\text{KdV}_1 \quad : \quad \partial_{T^1} u = \{u, \mathcal{H}_1\}_{\text{LM}} = \{u, \frac{1}{2} u^2\}_{\text{LM}} = (\partial_X^3 + 4u \partial_X + 2u_X) u$$

The Hamiltonian representation of an integrable hierarchy of PDEs, such as the KdV hierarchy also consists of an infinite family of commuting functionals. How can one

characterise hydrodynamic-type functionals that Poisson commute?

Lemma 8. [64, 35] *Let \mathcal{H} and \mathcal{F} define hydrodynamic-type functionals as in (3.1.9). Then*

$$\{\mathcal{H}, \mathcal{F}\} = 0 \Leftrightarrow g^{-1}(\nabla_\alpha dh, \nabla_\beta df) = g^{-1}(\nabla_\beta dh, \nabla_\alpha df), \quad (3.1.12)$$

where ∇ denotes the covariant derivative for the metric g .

Proof. In light of the differential-geometric characterisation of the Jacobi identity given by Dubrovin and Novikov, we can give a coordinate-free re-interpretation of the definition (3.1.8),

$$\{\mathcal{H}, \mathcal{F}\} = \int_{S^1} g^{-1}(dh, \nabla_X df) dX, \quad (3.1.13)$$

where $\nabla_X := \frac{\partial v^\alpha}{\partial X} \nabla_\alpha$. Therefore,

$$\{\mathcal{H}, \mathcal{F}\} = 0 \Leftrightarrow \int_{S^1} \frac{\partial v^\sigma}{\partial X} g^{-1}(dh, \nabla_\sigma df) dX = 0 =: \int_{S^1} v^* \theta.$$

Because exterior differentiation commutes with the pull-back map, the one-form $v^* \theta := \frac{\partial v^\sigma}{\partial X} g^{-1}(dh, \nabla_\sigma df) dX \in \Gamma(T^*S^1, S^1)$ integrates to zero on each loop (locally, or for \mathcal{M} simply connected) iff $d\theta = 0$, or in the flat coordinates of the metric g iff

$$\nabla_\alpha g^{-1}(dh, \nabla_\beta df) = \nabla_\beta g^{-1}(dh, \nabla_\alpha df).$$

Because ∇ is the Levi-Civita connection for g , we also have $\nabla g^{-1} = 0$. Further we know that in order for the bracket (3.1.13) to satisfy the Jacobi identity ∇ is flat, i.e. $[\nabla_\alpha, \nabla_\beta] = 0$. Therefore

$$\begin{aligned} \nabla_\alpha g^{-1}(dh, \nabla_\beta df) &= g^{-1}(\nabla_\alpha dh, \nabla_\beta df) + g^{-1}(dh, \nabla_\alpha \nabla_\beta df) \\ &= g^{-1}(\nabla_\alpha dh, \nabla_\beta df) + g^{-1}(dh, \nabla_\beta \nabla_\alpha df) \\ &= \nabla_\beta g^{-1}(dh, \nabla_\alpha df) \\ &\Leftrightarrow g^{-1}(\nabla_\alpha dh, \nabla_\beta df) = g^{-1}(\nabla_\beta dh, \nabla_\alpha df). \end{aligned}$$

□

3.1.5 Bi-Hamiltonian Structures and Flat Pencils of Metrics

The theorem of Dubrovin and Novikov gives a way to construct Poisson structures from flat metrics. On a Frobenius manifold, we have not one but *two* flat metrics which, as we saw in Chapter 1 combine to form a flat pencil: $g - \lambda \eta$ is again a flat metric for all values of λ . What is the interpretation of this stronger statement in the context of Hamiltonian systems? The answer is that it is a bi-Hamiltonian system.

As we have seen, the KdV equation is Hamiltonian with respect to two different Poisson brackets,

$$\text{KdV}_n \quad : \quad \partial_{T^n} u = \{u, \mathcal{H}_{n+1}\}_{\text{GZF}} = \{u, \mathcal{H}_{n+1}\}_{\text{LM}}$$

It was first noted by Magri [?] that these Poisson structures have the remarkable property that the linear combination

$$\{\cdot, \cdot\}_{\text{LM}} - \lambda \{\cdot, \cdot\}_{\text{GZF}}$$

is again a Poisson bracket for any value of λ . This was one of the main motivating examples for the

Definition 15. A bi-Hamiltonian structure on a manifold \mathcal{M} is a 2-dimensional linear subspace in the space of Poisson structure on \mathcal{M} .

Although the notion of a bi-Hamiltonian structure makes the connection with Frobenius manifolds more rigid, the main question we will be concerned with answering in this thesis will be how a certain symmetry present at the level of Frobenius manifolds lifts to the hydrodynamic flows (3.1.10). Therefore it will suffice for our purposes to study just one of the Hamiltonian structures.

3.2 The Principal Hierarchy

3.2.1 Families of Commuting Functionals on Frobenius Manifolds

The Hamiltonian representation of the KdV hierarchy comprised a Poisson structure, together with an infinite family of conservation laws. On a Frobenius manifold we have not just one, but two candidates for the Poisson structure, but what about the infinite family of commuting functionals?

Lemma 9. Let $(\mathcal{M}, \circ, e, E, \eta)$ be a Frobenius manifold. Let $\{h_\alpha(\mathbf{t}; z) : \alpha = 1, \dots, N\}$ be a fundamental system of solutions to the system (1.3.4), and consider the power series expansion

$$h_\alpha(\mathbf{t}; \lambda) = \sum_{n \geq 0} \lambda^n h_{n, \alpha}(\mathbf{t}).$$

Then the functionals

$$\mathcal{H}_{n, \alpha} = \int_{S^1} h_{n, \alpha}(\mathbf{t}) dX \tag{3.2.1}$$

satisfy

$$\{\mathcal{H}_{n, \alpha}, \mathcal{H}_{m, \beta}\} = 0, \quad \text{for } \alpha, \beta = 1, \dots, N, \text{ and } n, m \geq 0, \tag{3.2.2}$$

where

$$\{\mathcal{H}_{n, \alpha}, \mathcal{H}_{m, \beta}\} = \int_{S^1} \eta^{-1} (dh_{n, \alpha}(\mathbf{t}), \nabla_X dh_{m, \beta}(\mathbf{t})) dX.$$

Proof. Owing to Lemma 8, we are to show that

$$\eta^{-1}(\nabla_{\sigma} dh_{n,\alpha}(\mathbf{t}), \nabla_{\kappa} dh_{m,\beta}(\mathbf{t})) = \eta^{-1}(\nabla_{\kappa} dh_{n,\alpha}(\mathbf{t}), \nabla_{\sigma} dh_{m,\beta}(\mathbf{t})).$$

In the flat coordinates $\{t^{\sigma} : \sigma = 1, \dots, N\}$ of the metric η , this reads

$$\eta^{\mu\nu} \left(\frac{\partial^2 h_{n,\alpha}}{\partial t^{\sigma} \partial t^{\mu}} \frac{\partial^2 h_{m,\beta}}{\partial t^{\kappa} \partial t^{\nu}} - \frac{\partial^2 h_{n,\alpha}}{\partial t^{\sigma} \partial t^{\nu}} \frac{\partial^2 h_{m,\beta}}{\partial t^{\kappa} \partial t^{\mu}} \right) = 0.$$

Using the recursion relation (1.3.8), this becomes

$$\lambda \eta^{\mu\nu} (c_{\sigma\mu}^{\rho} c_{\kappa\nu}^{\varepsilon} - c_{\sigma\nu}^{\rho} c_{\kappa\mu}^{\varepsilon}) \frac{\partial h_{n-1,\alpha}}{\partial t^{\rho}} \frac{\partial h_{m-1,\beta}}{\partial t^{\varepsilon}} = 0, \quad (3.2.3)$$

which is exactly the associativity property of the Frobenius algebras, or WDVV equations. \square

To summarise, given a Frobenius manifold we have both an infinite family of commuting functionals and a bi-Hamiltonian structure given by our pencil of flat metrics. From the point of view of our discussion above, we have all the ingredients of an integrable hierarchy of partial differential equations.

3.2.2 The Principal Hierarchy

Definition 16. The principal hierarchy of a Frobenius manifold $(\mathcal{M}, \circ, e, E, \eta)$ is defined by

$$\frac{\partial t^{\sigma}}{\partial T^{n,\alpha}} = \{t^{\sigma}, \mathcal{H}_{n,\alpha}\}, \quad (3.2.4)$$

where

$$\{\mathcal{H}_{n,\alpha}, \mathcal{H}_{m,\beta}\} = \int_{S^1} \eta^{-1}(dh_{n,\alpha}(\mathbf{t}), \nabla_X dh_{m,\beta}(\mathbf{t})) dX. \quad (3.2.5)$$

Note that the flows of the hierarchy commute,

$$\frac{\partial^2 t^{\sigma}}{\partial T^{n,\alpha} \partial T^{m,\beta}} = \frac{\partial^2 t^{\sigma}}{\partial T^{m,\beta} \partial T^{n,\alpha}}$$

due to the involutivity of the functionals and the Jacobi identity. To gain a more explicit formula for the flows of this hierarchy, note that the coordinate fields may be expressed as functionals using a Dirac-delta function,

$$t^{\sigma}(X) = \int_{S^1} t^{\sigma}(Y) \delta(X - Y) dY.$$

Hence,

$$\begin{aligned}
\frac{\partial t^\sigma}{\partial T^{n,\alpha}} &= \int_{S^1} \delta_\kappa^\sigma \delta(X-Y) \eta^{\kappa\mu} \partial_Y \left(\frac{\partial h_{n,\alpha}(\mathbf{t})}{\partial t^\mu} \right) dY, \\
&= \eta^{\sigma\mu} \partial_X \left(\frac{\partial h_{n,\alpha}(\mathbf{t})}{\partial t^\mu} \right), \\
&= \eta^{\sigma\mu} \frac{\partial^2 h_{n,\alpha}(\mathbf{t})}{\partial t^\mu \partial t^\nu} t_X^\nu, \\
&= \eta^{\sigma\mu} c_{\mu\nu}^\kappa \frac{\partial h_{n,\alpha}(\mathbf{t})}{\partial t^\kappa} t_X^\nu.
\end{aligned}$$

This makes it clear how the properties of the principal hierarchy, such as commutativity of the flows, depend on the algebraic structure present on the tangent spaces $T_t\mathcal{M}$. If you like, this is another manifestation of equation (3.2.3) present in the proof of Lemma 9.

Example 27 (Frobenius manifold for the KdV hierarchy). *If we re-scale both the space and time variables, $X \rightarrow \varepsilon X$, $T \rightarrow \varepsilon T$, $|\varepsilon| < 1$, the KdV equation becomes*

$$u_T = 6uu_X + \varepsilon^2 u_{XXX},$$

which is known as the small dispersion expansion of the KdV equation. Taking the limit $\varepsilon \rightarrow 0$, we obtain the dispersionless, or quasi-classical limit of the KdV equation,

$$u_T = 6uu_X, \tag{3.2.6}$$

known as the Burger's-Hopf equation. This is the simplest example of a nonlinear wave equation. In this limit, the behaviour of solutions changes dramatically: soliton solutions like (3.1.2) no longer exists, with solutions to (3.2.6) becoming multi-valued in finite time. It is the balance of the nonlinear uu_X term with the dispersion u_{XXX} term in the KdV equation that allow the existence of soliton solutions.

Having said this, the Jacobi identity for the Poisson brackets $\{ , \}_{\text{GZF}}$, $\{ , \}_{\text{LM}}$ and involutivity of the conservation laws are satisfied identically in ε . Hence the Burger's-Hopf equation (3.2.6) is bi-Hamiltonian, with respect to the Poisson brackets

$$\begin{aligned}
\{\mathcal{F}, \mathcal{H}\}_1 &= \{\mathcal{F}, \mathcal{H}\}_{\text{GZF}} \\
\{\mathcal{F}, \mathcal{H}\}_2 &= \lim_{\varepsilon \rightarrow 0} \int_{S^1} \frac{\delta \mathcal{F}}{\delta u} (\varepsilon^2 \partial_X^3 + 4u \partial_X + 2u_X) \left(\frac{\delta \mathcal{H}}{\delta u} \right) dX \quad " = " \quad \lim_{\varepsilon \rightarrow 0} \{ \cdot, \cdot \}_{\text{LM}},
\end{aligned}$$

and it admits infinitely many conserved quantities,

$$\widehat{\mathcal{H}}_n(u) := \lim_{\varepsilon \rightarrow 0} \mathcal{H}_n(u, \varepsilon u_X, \varepsilon^2 u_{XX}, \dots), \quad n \geq 0.$$

Recall that for the trivial Frobenius manifold,

$$F = \frac{1}{6}t^3, \quad E = t \frac{\partial}{\partial t},$$

the deformed flat coordinate was

$$h(\mathbf{t}; \lambda) = \sum_{n \geq 0} \lambda^n \frac{t^{n+1}}{(n+1)!},$$

which leads to the infinite family of functionals

$$\mathcal{H}_n = \frac{1}{(n+1)!} \int_{S^1} t^{n+1} dX, \quad n \geq 0.$$

The first Hamiltonian structure for this manifold is given by

$$\{\mathcal{H}_n, \mathcal{H}_m\} = \int_{S^1} dh_n(\mathbf{t}) \partial_X dh_m(\mathbf{t}) dX,$$

which leads to the flows

$$\{t, \mathcal{H}_n\} = \frac{1}{n!} t^n t_X, \quad n \geq 0,$$

which we recognise as the (slightly re-scaled) dispersionless limit of the KdV hierarchy. For example, for $n = 1$ recover Burger's equation (3.2.6). The intersection form is given by $g = t$, and so the second Hamiltonian structure is given by

$$\{\mathcal{H}_n, \mathcal{H}_m\}_2 = \int_{S^1} dh_n(\mathbf{t}) \left(t \partial_X + \frac{1}{2} t_X \right) dh_m(\mathbf{t}) dX.$$

This second Hamiltonian structure is the dispersionless limit of the Lenard-Magri bracket.

It may be shown (see, for example [16], [17]), that the principal hierarchy possesses a *tau-structure*: Let $\{v^\sigma(X, T) : \sigma = 1, \dots, N\}$ be a solution of the principal hierarchy. Then all the Hamiltonian densities are given by

$$h_{n,\alpha}(\mathbf{v}) = \frac{\partial^2 \log \tau(T)}{\partial X \partial T^{n,\alpha}}, \quad \alpha = 1, \dots, N, \quad n \geq 0. \quad (3.2.7)$$

It was proved by Dubrovin in [17] that on the small phase space $\mathcal{M} := \{T^{\alpha,p} = 0, \text{ for } p \geq 0; T^{\alpha,0} \equiv t^\alpha\}$, the logarithm of the tau-function coincides with the genus zero pre-potential of the Frobenius manifold,

$$\log \tau|_{\mathcal{M}} = F(\mathbf{v}). \quad (3.2.8)$$

In particular, for $n = 0$, the solution itself is given by (we identify the time $T^{0,1} = X$ with the spatial variable)

$$v^\sigma = \frac{\partial^2 \log \tau}{\partial X \partial T^{0,\sigma}}.$$

This structure is important to an approach given by Dubrovin & Zhang [23] to re-constructing a fully dispersive hierarchy, like the KdV hierarchy, from its dispersionless limit. In some cases, all the information required to re-construct the full hierarchy is contained in the dispersionless limit, with this re-construction being performed by a so-called *Muir map*, which takes the form

$$v^\sigma \mapsto \sum_{k \geq 0} F_k^\sigma(\mathbf{v}; \mathbf{v}_X, \dots, \mathbf{v}_{kX}), \quad \text{for } \sigma = 1, \dots, N.$$

One should keep in mind here moving from the genus zero to the full Gromov-Witten potential of a symplectic manifold. From the point of view of the Hamiltonian representations considered here, such a reconstruction is possible when the Poisson cohomology of the hierarchy vanishes, which is true for any Frobenius manifold [24].

Chapter 4

The Inversion Symmetry and Principal Hierarchies

4.1 The Inversion Symmetry in Canonical Coordinates

A natural question has been raised. Suppose one has two Frobenius manifolds linked by the inversion symmetry, or indeed a modular Frobenius manifold. How does this symmetry lift to the principal hierarchy of the Frobenius manifolds? The answer to this question will come in two steps. Firstly, we must understand how the Hamiltonians of the principal hierarchy are transformed. Recall that these were the successive approximations to the deformed flat coordinates of the Frobenius manifold. Secondly, we must understand the action on the Poisson structure used to define the Hamiltonian vector fields on the loop space of the manifold. A key technical observation will be that in the canonical coordinates the inversion symmetry acts as a conformal transformation of the metric.

Let us begin by collecting some technical results. Throughout this chapter and the next, $t_1 := \eta_{1\alpha} t^\alpha = t^N$ is the Egoroff potential for the metric η .

Proposition 13. *Under the inversion symmetry, the canonical coordinates are fixed,*

$$u_i = \hat{u}_i, \quad \text{for } i = 1, \dots, N.$$

Proof. We begin by observing that the inversion symmetry acts as a conformal transformation of both metrics. This follows from the transformation properties of the structure functions (1.5.5):

$$c_{\alpha\beta\gamma} = \hat{t}_1^{-2} \frac{\partial \hat{t}^\sigma}{\partial t^\alpha} \frac{\partial \hat{t}^\kappa}{\partial t^\beta} \frac{\partial \hat{t}^\lambda}{\partial t^\gamma} \hat{c}_{\sigma\kappa\lambda} \Rightarrow \eta_{\alpha\beta} = \hat{t}_1^{-2} \frac{\partial \hat{t}^\sigma}{\partial t^\alpha} \frac{\partial \hat{t}^\kappa}{\partial t^\beta} \hat{\eta}_{\sigma\kappa} \Rightarrow \eta^{\alpha\beta} = \hat{t}_1^2 \frac{\partial t^\alpha}{\partial \hat{t}^\sigma} \frac{\partial t^\beta}{\partial \hat{t}^\kappa} \hat{\eta}^{\sigma\kappa}.$$

From this it follows that

$$c_\gamma^{\alpha\beta} = \hat{t}_1^2 \frac{\partial \hat{t}^\alpha}{\partial t^\sigma} \frac{\partial \hat{t}^\beta}{\partial t^\kappa} \frac{\partial t^\nu}{\partial \hat{t}^\gamma} \hat{c}_\nu^{\sigma\kappa} \Rightarrow g^{\alpha\beta} = \hat{t}_1^2 \frac{\partial t^\alpha}{\partial \hat{t}^\sigma} \frac{\partial t^\beta}{\partial \hat{t}^\kappa} \hat{g}^{\sigma\kappa}.$$

Recall (1.4.18) that the canonical coordinates may be defined as the roots of the characteristic equation

$$\det \left(g^{\alpha\beta}(\mathbf{t}) - u \eta^{\alpha\beta} \right) = 0.$$

It follows from the transformation properties of the contravariant metrics that the roots of this equation are invariant under the inversion symmetry. Hence the canonical coordinates are invariant, up to re-ordering. □

Proposition 14. *In the canonical coordinate system, the inversion symmetry acts as a conformal transformation of the metric η .*

Proof. Recall that in the canonical coordinate system the metric is Egoroff, with potential

$t_1(u)$:

$$\eta = \sum_{i=1}^N \partial_i t_1(u) du_i \otimes du_i.$$

The inversion symmetry acts as a Möbius transformaion of the Egoroff potential:

$$t_1 \mapsto \hat{t}_1 = -\frac{1}{t_1} \quad \Rightarrow \quad \hat{\eta}_{ii} = \frac{1}{t_1^2} \eta_{ii}.$$

□

Example 28. Recall for the Frobenius manifold defined by the prepotential and Euler field

$$F = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2} t^1 (t^2)^2 - \frac{(t^2)^4}{16} \gamma(t^3); \quad E = t^1 \frac{\partial}{\partial t^1} + \frac{1}{2} t^2 \frac{\partial}{\partial t^2},$$

the canonical coordinates were given by

$$u_i = t^1 + \frac{1}{2} (t^2)^2 \omega_i(t^3), \quad \text{for } i = 1, 2, 3,$$

where $\{\omega_1, \omega_2, \omega_3\}$ satisfy the Halphen system (2.3.3). Solutions to the Halphen system have the modularity properties [43]

$$\omega_i\left(-\frac{1}{\tau}\right) = \tau^2 \omega_i(\tau) - \tau, \quad \omega_i(\tau + 1) = \omega_i(\tau), \quad \text{for } i = 1, 2, 3.$$

Using this fact, we have

$$u_i(\mathbf{t}(\hat{\mathbf{t}})) = \hat{t}_1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3} + \frac{1}{2} \left(-\frac{\hat{t}^2}{\hat{t}^3}\right)^2 \omega_i\left(-\frac{1}{\hat{t}^3}\right) = \hat{t}^1 + \frac{1}{2} (\hat{t}^2)^2 \omega_i(\hat{t}^3) = u_i(\hat{\mathbf{t}}).$$

The rotation coefficients may be re-cast in terms of the Egoroff potential:

$$\beta_{ij}(u) = \frac{\partial_j \sqrt{\eta_{ii}(u)}}{\sqrt{\eta_{jj}(u)}} = \frac{1}{2} \frac{\partial_i \partial_j t_1}{\sqrt{\partial_i t_1 \partial_j t_1}},$$

where we have dropped the explicit dependence $t_1 = t_1(u)$. The inversion symmetry acts by a shift on the rotation coefficients: If $\hat{\beta}_{ij}$ denote the rotation coefficients of the inverted Frobenius manifold, then they are related to those of the original manifold by

$$\hat{\beta}_{ij}(u) = \beta_{ij}(u) - \frac{1}{t_1(u)} \sqrt{\partial_i t_1 \partial_j t_1}.$$

Using standard formulae of Riemannian geometry (see Section 1.4), one finds the follow-

ing formulae for the non-zero Christoffel symbols:

$$\Gamma_{ij}^i = \beta_{ij} \sqrt{\frac{\partial_j t_1}{\partial_i t_1}}, \quad \Gamma_{ii}^j = -\beta_{ij} \sqrt{\frac{\partial_i t_1}{\partial_j t_1}}, \quad \text{for } i \neq j,$$

$$\Gamma_{ii}^i = -\sum_{k \neq i} \beta_{ik} \sqrt{\frac{\partial_k t_1}{\partial_i t_1}}.$$

Putting this together with the transformation properties of the rotation coefficients immediately gives the action of the inversion symmetry on the Christoffel symbols:

$$\begin{aligned} \Gamma_{ij}^i &= \hat{\Gamma}_{ij}^i - \frac{1}{\hat{t}_1} \partial_j \hat{t}_1, \quad i \neq j \\ \Gamma_{ii}^j &= \hat{\Gamma}_{ii}^j + \frac{1}{\hat{t}_1} \partial_i \hat{t}_1, \quad i \neq j, \\ \Gamma_{ii}^i &= \hat{\Gamma}_{ii}^i + \frac{1}{\hat{t}_1} \sum_{k \neq i} \partial_k \hat{t}_1 = \hat{\Gamma}_{ii}^i - \frac{1}{\hat{t}_1} \partial_i \hat{t}_1, \end{aligned} \tag{4.1.1}$$

where the last inequality follows from the identity $e(t_1(u)) = \sum_{i=1}^N \partial_i t_1(u) = 0$.

4.2 Inversion Symmetry and Principal Hierarchies

We aim now to construct an ansatz for how the inversion symmetry acts on the deformed flat coordinates. Let us take as motivation our canonical example of a modular Frobenius manifold.

Example 29. Recall that for the Frobenius manifold defined by the data

$$F = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 + \frac{t_2^4}{16} \gamma(t_3), \quad E = t_1 \frac{\partial}{\partial t_1} + \frac{1}{2} \frac{\partial}{\partial t_2},$$

where $\gamma(t_3)$ satisfies the Chazy equation,

$$\gamma''' = 6\gamma\gamma'' - 9(\gamma')^2,$$

we have the recursion relation

$$\frac{\partial^2 h_\sigma^{(n)}}{\partial t^\mu \partial t^\nu} = c_{\mu\nu}^\varepsilon \frac{\partial h_\sigma^{(n-1)}}{\partial t^\varepsilon}, \quad \text{for } \mu, \nu, \sigma = 1, 2, 3, \quad n \geq 0,$$

subject to $h_\sigma^{(0)} = t_\sigma$. Solving this equation recursively gives the solutions

$$\begin{aligned}
 h_1(\mathbf{t}; \lambda) &= t^3 + \lambda \left\{ t^1 t^3 + \frac{1}{2} (t^2)^2 \right\} \\
 &\quad + \lambda^2 \left\{ -\frac{1}{16} t^3 (t^2)^4 \gamma'(t^3) - \frac{1}{8} (t^2)^4 \gamma(t^3) + \frac{1}{2} t^1 (t^2)^2 + \frac{1}{2} (t^1)^2 t^3 \right\} + \mathcal{O}(\lambda^3), \\
 h_2(\mathbf{t}; \lambda) &= t^2 + \lambda \left\{ t^1 t^2 - \frac{1}{4} (t^2)^3 \gamma(t^3) \right\} \\
 &\quad + \lambda^2 \left\{ -\frac{1}{20} (t^2)^5 \gamma'(t^3) + \frac{9}{160} (t^2)^5 \gamma(t^3)^2 - \frac{1}{4} t^1 (t^2)^3 \gamma(t^3) + \frac{1}{2} (t^1)^2 t^2 \right\} + \mathcal{O}(\lambda^3), \\
 h_3(\mathbf{t}; \lambda) &= t^1 + \lambda \left\{ \frac{1}{2} (t^1)^2 - \frac{1}{16} (t^2)^4 \gamma'(t^3) \right\} \\
 &\quad + \lambda^2 \left\{ -\frac{1}{480} (t^2)^6 \gamma''(t^3) + \frac{1}{80} (t^2)^6 \gamma(t^3) \gamma'(t^3) - \frac{1}{16} t^1 (t^2)^4 \gamma'(t^3) + \frac{1}{6} (t^1)^3 \right\} + \mathcal{O}(\lambda^3).
 \end{aligned}$$

Let us compute how some of the coefficients transform. For example,

$$\begin{aligned}
 h_{0,1}(\mathbf{t}) &= t^3 \Rightarrow h_{0,1} \left(\hat{t}^1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3}, -\frac{\hat{t}^2}{\hat{t}^3}, -\frac{1}{\hat{t}^3} \right) = -\frac{1}{\hat{t}^3} = -\frac{1}{\hat{t}^3} h_{-1,3}(\hat{t}^1, \hat{t}^2, \hat{t}^3); \\
 h_{0,2}(\mathbf{t}) &= t^2 \Rightarrow h_{0,2} \left(\hat{t}^1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3}, -\frac{\hat{t}^2}{\hat{t}^3}, -\frac{1}{\hat{t}^3} \right) = -\frac{\hat{t}^2}{\hat{t}^3} = -\frac{1}{\hat{t}^3} h_{0,2}(\hat{t}^1, \hat{t}^2, \hat{t}^3); \\
 h_{0,3}(\mathbf{t}) &= t^1 \Rightarrow h_{0,3} \left(\hat{t}^1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3}, -\frac{\hat{t}^2}{\hat{t}^3}, -\frac{1}{\hat{t}^3} \right) = \hat{t}^1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3} = \frac{1}{\hat{t}^3} h_{1,1}(\hat{t}^1, \hat{t}^2, \hat{t}^3); \\
 h_{1,1}(\mathbf{t}) &= t^1 t^3 + \frac{1}{2} (t^2)^2 \Rightarrow h_{1,1} \left(\hat{t}^1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3}, -\frac{\hat{t}^2}{\hat{t}^3}, -\frac{1}{\hat{t}^3} \right) = \left(\hat{t}^1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3} \right) \left(-\frac{1}{\hat{t}^3} \right) + \frac{1}{2} \left(\frac{\hat{t}^2}{\hat{t}^3} \right)^2 \\
 &= -\frac{\hat{t}^1}{\hat{t}^3} - \frac{1}{2} \left(\frac{\hat{t}^2}{\hat{t}^3} \right)^2 + \frac{1}{2} \left(\frac{\hat{t}^2}{\hat{t}^3} \right)^2 = -\frac{\hat{t}^1}{\hat{t}^3} = -\frac{1}{\hat{t}^3} h_{0,3}(\hat{t}^1, \hat{t}^2, \hat{t}^3); \\
 h_{1,2}(\mathbf{t}) &= t^1 t^2 - \frac{1}{4} (t^2)^3 \gamma(t^3) \Rightarrow h_{1,2} \left(\hat{t}^1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3}, -\frac{\hat{t}^2}{\hat{t}^3}, -\frac{1}{\hat{t}^3} \right) = \left(\hat{t}^1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3} \right) \left(-\frac{\hat{t}^2}{\hat{t}^3} \right) + \frac{1}{4} \left(\frac{\hat{t}^2}{\hat{t}^3} \right)^3 \gamma \left(-\frac{1}{\hat{t}^3} \right) \\
 &= -\frac{\hat{t}^1 \hat{t}^2}{\hat{t}^3} + \frac{(\hat{t}^2)^3}{4\hat{t}^3} \left\{ \frac{1}{(\hat{t}^3)^2} \gamma \left(-\frac{1}{\hat{t}^3} \right) - \frac{2}{\hat{t}^3} \right\} \\
 &= -\frac{1}{\hat{t}^3} \left(\hat{t}^1 \hat{t}^2 - \frac{1}{4} (\hat{t}^2)^3 \gamma(\hat{t}^3) \right) = -\frac{1}{\hat{t}^3} h_{1,2}(\hat{t}^1, \hat{t}^2, \hat{t}^3); \\
 h_{1,3}(\mathbf{t}) &= \frac{1}{2} (t^1)^2 - \frac{1}{16} (t^2)^4 \gamma'(t^3) \Rightarrow h_{1,3} \left(\hat{t}^1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3}, -\frac{\hat{t}^2}{\hat{t}^3}, -\frac{1}{\hat{t}^3} \right) = \frac{1}{2} \left(\hat{t}^1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3} \right)^2 - \frac{1}{16} \left(\frac{\hat{t}^2}{\hat{t}^3} \right)^4 \gamma' \left(-\frac{1}{\hat{t}^3} \right) \\
 &= \frac{1}{\hat{t}^3} \left(\frac{1}{2} (\hat{t}^1)^2 \hat{t}^3 + \frac{1}{2} \hat{t}^1 (\hat{t}^2)^2 \right) - \frac{(\hat{t}^2)^4}{16} \left\{ \frac{1}{(\hat{t}^3)^4} \gamma' \left(-\frac{1}{\hat{t}^3} \right) - \frac{2}{(\hat{t}^3)^2} \right\} \\
 &= \frac{1}{\hat{t}^3} \left(\frac{1}{2} (\hat{t}^1)^2 \hat{t}^3 + \frac{1}{2} \hat{t}^1 (\hat{t}^2)^2 - \frac{1}{16} \hat{t}^3 (\hat{t}^2)^4 \gamma'(\hat{t}^3) - \frac{1}{8} (\hat{t}^2)^4 \gamma(\hat{t}^3) \right) = \frac{1}{\hat{t}^3} h_{2,1}(\hat{t}^1, \hat{t}^2, \hat{t}^3).
 \end{aligned}$$

Note the use of the modularity properties of the solution to Chazy's differential equation and its derivative. One can continue to compute explicitly the transformation properties

of higher order approximations, but the calculations rapidly become more involved.

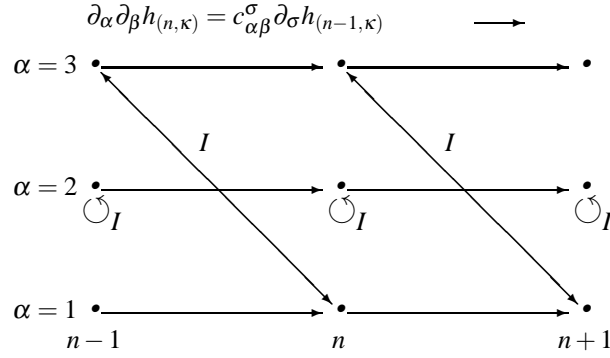
In general, we observe the following pattern:

$$h_{n,\alpha}(\mathbf{t}(\hat{\mathbf{t}})) = h_{n,\alpha} \left(\hat{t}^1 + \frac{(\hat{t}^2)^2}{2\hat{t}^3}, -\frac{\hat{t}^2}{\hat{t}^3}, -\frac{1}{\hat{t}^3} \right) = \pm \frac{1}{\hat{t}^3} h_{\tilde{n},\tilde{\alpha}}(\hat{\mathbf{t}}),$$

where

$$\tilde{n} = \begin{cases} n+1, & \text{if } \alpha = 3, \\ n, & \text{if } \alpha = 2, \\ n-1, & \text{if } \alpha = 1, \end{cases} \quad \tilde{\alpha} = \begin{cases} 1, & \text{if } \alpha = 3, \\ \alpha, & \text{if } \alpha = 2, \\ 3, & \text{if } \alpha = 1, \end{cases} \quad \pm = \begin{cases} +, & \text{if } \alpha = 3, \\ -, & \text{else.} \end{cases}$$

This can be summarised diagrammatically:



We take this example as motivation for the more general ansatz that extends to Frobenius manifolds that do not lie at fixed points of the inversion symmetry (in general it is not true that $h_{n,\alpha}(\hat{\mathbf{t}}) = \hat{h}_{n,\alpha}(\hat{\mathbf{t}})$, as in the above example). The results of the next Proposition were obtained independently in [41]. Recently a third, and rather different proof, has appeared in [26].

Proposition 15. *The inversion symmetry acts on the deformed flat coordinates of a semi-simple Frobenius manifold as*

$$h_{n,\alpha}(\mathbf{t}(\hat{\mathbf{t}})) = \pm \frac{1}{\hat{t}^3} \hat{h}_{\tilde{n},\tilde{\alpha}}(\hat{\mathbf{t}}), \quad (4.2.1)$$

where

$$\tilde{n} = \begin{cases} n+1, & \text{if } \alpha = N, \\ n, & \text{if } \alpha \neq 1, N, \\ n-1, & \text{if } \alpha = 1, \end{cases} \quad \tilde{\alpha} = \begin{cases} 1, & \text{if } \alpha = N, \\ \alpha, & \text{if } \alpha \neq 1, N, \\ N, & \text{if } \alpha = 1, \end{cases} \quad \pm = \begin{cases} +, & \text{if } \alpha = N, \\ -, & \text{else.} \end{cases} \quad (4.2.2)$$

Proof. Firstly, we need to show that if $h_{n,\alpha}(t^1, \dots, t^N)$ satisfies the recursion relation

$$\frac{\partial^2 h_{n,\alpha}}{\partial t^\mu \partial t^\nu} = c_{\mu\nu}^\sigma \frac{\partial h_{n-1,\alpha}}{\partial t^\sigma}$$

then $\mathfrak{h}_{n,\alpha}(\hat{t}^1, \dots, \hat{t}^N) := \hat{t}^N h_{n,\alpha}\left(\frac{\hat{t}_\alpha \hat{t}^\sigma}{2\hat{t}^N}, -\frac{\hat{t}^2}{\hat{t}^N}, \dots, -\frac{\hat{t}^{N-1}}{\hat{t}^N}, -\frac{1}{\hat{t}^N}\right)$ satisfies the recursion relation

$$\frac{\partial^2 \mathfrak{h}_{n,\alpha}}{\partial \hat{t}^\mu \partial \hat{t}^\nu} = \hat{c}_{\mu\nu}^\sigma \frac{\partial \mathfrak{h}_{n-1,\alpha}}{\partial \hat{t}^\sigma}. \quad (4.2.3)$$

Secondly, we must verify the identification of the labels, which will be done using their homogeneity properties (1.3.9). We begin by noting that in the canonical coordinate system the recursion relation becomes

$$\nabla_i \nabla_j h_{n,\alpha} = \delta_{ij} \nabla_j h_{n-1,\alpha},$$

where ∇ denotes the covariant derivative of the Levi-Civita connection for the metric η . In particular, the first statement we need to prove (4.2.3) reads

$$\hat{\nabla}_i \frac{\partial}{\partial u_j} (\hat{t}^N h_{n,\alpha}) = \delta_{ij} \frac{\partial}{\partial u_j} (\hat{t}^N h_{n-1,\alpha}). \quad (4.2.4)$$

To this end, let us compute how the covariant derivatives of two one-forms are related via the inversion symmetry. This is a direct calculation using the transformation properties of the Christoffel symbols derived above. Let $\phi = \sum_i \phi_i du_i \in \Gamma(T^*M, M)$. Because of the index dependence of the transformation properties of the Christoffel symbols we decompose the computation. Suppose $i \neq j$. Then

$$\begin{aligned} \nabla_j \phi_i &= \partial_j \phi_i - \sum_{s=1}^N \Gamma_{ji}^s \phi_s = \partial_j \phi_i - \Gamma_{ji}^j \phi_j - \Gamma_{ji}^i \phi_i \\ &= \partial_j \phi_i - \left(\hat{\Gamma}_{ji}^j - \partial_i \log \hat{t}_1 \right) \phi_j - \left(\hat{\Gamma}_{ji}^i - \partial_j \log \hat{t}_1 \right) \phi_i \\ &= \hat{\nabla}_j \phi_i + (\partial_i \log \hat{t}_1) \phi_j + (\partial_j \log \hat{t}_1) \phi_i. \end{aligned} \quad (4.2.5)$$

If $i = j$, we have

$$\begin{aligned} \nabla_i \phi_i &= \partial_i \phi_i - \sum_{s=1}^N \Gamma_{ii}^s \phi_s = \partial_i \phi_i - \Gamma_{ii}^i \phi_i = \sum_{s \neq i} \Gamma_{ii}^s \phi_s \\ &= \partial_i \phi_i - \left(\hat{\Gamma}_{ii}^i - \partial_i \log \hat{t}_1 \right) \phi_i - \sum_{s \neq i} \left(\hat{\Gamma}_{ii}^s + \partial_i \log \hat{t}_1 \right) \phi_s \\ &= \hat{\nabla}_i \phi_i + (\partial_i \log \hat{t}_1) \phi_i - (\partial_i \log \hat{t}_1) \sum_{s \neq i} \phi_s. \end{aligned} \quad (4.2.6)$$

Consider the left hand side of (4.2.4). If $i \neq j$, we have

$$\begin{aligned}
\hat{\nabla}_i \partial_j (\hat{t}_1 h_{n,\alpha}) &= \hat{\nabla}_i (\hat{\eta}_{jj} h_{n,\alpha} + \hat{t}_1 \partial_j h_{n,\alpha}) \\
&= \hat{\eta}_{jj} \partial_i h_{n,\alpha} + \partial_i \hat{t}_1 \partial_j h_{n,\alpha} + \hat{t}_1 \hat{\nabla}_i \partial_j h_{n,\alpha} \quad \text{as } \hat{\nabla} \hat{\eta} = 0, \\
&= \frac{1}{\hat{t}_1^2} \eta_{jj} \partial_i h_{n,\alpha} + \frac{1}{\hat{t}_1^2} \eta_{ii} \partial_j h_{n,\alpha} - \frac{1}{t_1} \left\{ \nabla_i \partial_j h_{n,\alpha} + \frac{1}{t_1} \eta_{ii} \partial_j h_{n,\alpha} + \frac{1}{t_1} \eta_{jj} \partial_i h_{n,\alpha} \right\} \\
&= -\frac{1}{t_1} \nabla_i \partial_j h_{n,\alpha} = 0, \quad \text{if } i \neq j,
\end{aligned}$$

because the $h_{n,\alpha}$ satisfy the recursion relation of the uninverted manifold. If, on the other hand $i = j$, the left hand side of (4.2.4) reads

$$\begin{aligned}
\hat{\nabla}_i \partial_i (\hat{t}_1 h_{n,\alpha}) &= \hat{\nabla}_i (\partial_i \hat{t}_1 h_{n,\alpha} + \hat{t}_1 \partial_i h_{n,\alpha}) \\
&= 2 \partial_i \hat{t}_1 \partial_i h_{n,\alpha} + \hat{t}_1 \hat{\nabla}_i \partial_i h_{n,\alpha} \\
&= 2 \partial_i \hat{t}_1 \partial_i h_{n,\alpha} + \hat{t}_1 \left\{ \nabla_i \partial_i h_{n,\alpha} + \partial_i \log t_1 \partial_i h_{n,\alpha} - \partial_i \log t_1 \sum_{s \neq i} \partial_s h_{n,\alpha} \right\} \\
&= \frac{\partial_i t_1}{t_1^2} \partial_i h_{n,\alpha} + \hat{t}_1 \nabla_i (\partial_i h_{n,\alpha}) + \frac{1}{t_1^2} \partial_i t_1 (h_{n-1,\alpha} - \partial_i h_{n,\alpha}) \\
&= \hat{t}_1 \nabla_i (\partial_i h_{n,\alpha}) + (\partial_i \hat{t}_1) h_{n-1,\alpha},
\end{aligned}$$

which is equal to the right hand side of the equation (4.2.4) in the case where $i = j$. Note that we have made use of the simplified recursion,

$$e(h_{n,\alpha}) = h_{n-1,\alpha} \quad \Rightarrow \quad \sum_{s \neq i} \partial_s h_{n,\alpha} = h_{n-1,\alpha} - \partial_i h_{n,\alpha}.$$

In conclusion, the functions $\frac{1}{\hat{t}_1} h_{n,\alpha} \left(\frac{\hat{t}_\sigma \hat{t}^\sigma}{2\hat{t}^N}, -\frac{\hat{t}^2}{\hat{t}^N}, \dots, -\frac{\hat{t}^{N-1}}{\hat{t}^N}, -\frac{1}{\hat{t}^N} \right)$ satisfy the recursion relations of the inverted manifold.

To identify the labels (4.2.2), we are to show

$$\mathcal{L}_E \left\{ h_{n,\alpha} \left(\frac{\hat{t}_\sigma \hat{t}^\sigma}{2\hat{t}^N}, -\frac{\hat{t}^2}{\hat{t}^N}, \dots, -\frac{\hat{t}^{N-1}}{\hat{t}^N}, -\frac{1}{\hat{t}^N} \right) \right\} = \mathcal{L}_{\hat{E}} \left\{ \frac{1}{\hat{t}_1} \hat{h}_{\tilde{n},\tilde{\alpha}} \right\}. \quad (4.2.7)$$

This follows from how the spectrum of the Frobenius manifold is mapped under the inversion symmetry. By definition of the inversion symmetry, we have

$$\hat{\mu}_1 = 1 - \mu_N, \quad \mu_i = \hat{\mu}_i, \text{ for } i \neq 1, N, \quad \hat{\mu}_N = \mu_1 + 1, \quad \hat{d} = 2 - d.$$

From this one can compute the matrices $(R_r)^\mu_\nu$ in terms of $(\hat{R}_r)^\mu_\nu$. Recall that these matri-

ces were defined by (1.3.9) on page 15. For example, let $\alpha = i \neq 1, N$. We have

$$\begin{aligned}
\mathcal{L}_E(h_{n,i}) &= (d_{N-i+1} + n)h_{n,i} + \sum_{r=1}^n (R_r)_i^\sigma h_{n-r,\sigma} \\
&= (\hat{d}_{N-i+1} - \hat{d}_N + n)h_{n,i} + \sum_{r=1}^n (R_r)_i^N h_{n-r,N} + \sum_{r=1}^n (R_r)_i^1 h_{n-r,1} + \sum_{r=1}^n (R_r)_i^j h_{n-r,j}, \\
&= (\hat{d}_{N-i+1} - \hat{d}_N + n)h_{n,i} + \sum_{r=1}^n (\hat{R}_{r-1})_i^1 h_{n-r,N} + \sum_{r=1}^n (\hat{R}_{r+1})_i^N h_{n-r,1} + \sum_{r=1}^n (\hat{R}_r)_i^j h_{n-r,j}, \\
&= (\hat{d}_{N-i+1} - \hat{d}_N + n)h_{n,i} + \sum_{r=1}^n (\hat{R}_r)_i^1 h_{n-r+1,N} + \sum_{r=1}^n (\hat{R}_r)_i^N h_{n-r-1,1} + \sum_{r=1}^n (\hat{R}_r)_i^j h_{n-r,j}.
\end{aligned}$$

On the other hand, (recall that we claim $\tilde{n} = n$, if $\alpha = i \neq 1, N$)

$$\begin{aligned}
\mathcal{L}_{\hat{E}}\left(\frac{1}{\hat{t}_1}\hat{h}_{n,i}\right) &= (\hat{d}_{N-i+1} + n - \hat{d}_N)\frac{1}{\hat{t}_1}\hat{h}_{n,i} + \frac{1}{\hat{t}_1}\sum_{r=1}^n (\hat{R}_r)_i^\sigma \hat{h}_{n-r,\sigma} \\
&= (\hat{d}_{N-i+1} - \hat{d}_N + n)\frac{1}{\hat{t}_1}\hat{h}_{n,i} + \frac{1}{\hat{t}_1}\sum_{r=1}^n (\hat{R}_r)_i^N \hat{h}_{n-r,N} + \frac{1}{\hat{t}_1}\sum_{r=1}^n (\hat{R}_r)_i^1 \hat{h}_{n-r,1} + \frac{1}{\hat{t}_1}\sum_{r=1}^n (\hat{R}_r)_i^j \hat{h}_{n-r,j}.
\end{aligned}$$

Note then that our ansatz then satisfies the equality (4.2.7). Analogous calculations are possible for $\alpha = 1$ and $\alpha = N$ and will be omitted.

Finally the identification of the signs is straightforward: If $h_{k,\alpha}(\mathbf{t}(\hat{\mathbf{t}})) = \pm \frac{1}{\hat{t}_1}\hat{h}_{\tilde{k},\tilde{\alpha}}(\hat{\mathbf{t}})$, then $h_{k+1,\alpha}(\mathbf{t}(\hat{\mathbf{t}})) = \pm \frac{1}{\hat{t}_1}\hat{h}_{\tilde{k}+1,\tilde{\alpha}}(\hat{\mathbf{t}})$, i.e. the recursion (1.3.8) respects signs. Thus the signs filter up from those present at the level of the Casimirs, which are as stated in the proposition. \square

One is now naturally led to consider how this action lifts to the flows. As with Proposition 15, this result was also obtained independently by Zhang, et. al. [41].

Proposition 16. *The inversion symmetry acts on the principal hierarchy of a semi-simple Frobenius manifold by*

$$\frac{\partial}{\partial T^{n,\alpha}} = \pm \hat{t}_1 \frac{\partial}{\partial \hat{T}^{\tilde{n},\tilde{\alpha}}} \mp \hat{h}_{\tilde{n}-1,\tilde{\alpha}} \frac{\partial}{\partial X} \quad (4.2.8)$$

where the definitions of \tilde{n} , $\tilde{\alpha}$, and \pm are as above.

Proof. Most of the hard work has been done in the previous proposition. Recall that the principal hierarchy in flat coordinates takes the form

$$\frac{\partial t^\sigma}{\partial T^{n,\alpha}} = \eta^{\sigma\varepsilon} c_{\varepsilon\nu}^\lambda \frac{\partial h_{n-1,\alpha}}{\partial t^\lambda} \frac{\partial t^\nu}{\partial X}.$$

Then it follows that in the canonical coordinates it takes the form

$$\frac{\partial u_r}{\partial T^{n,\alpha}} = \eta^{rr} \frac{\partial h_{n-1,\alpha}}{\partial u_r} \frac{\partial u_r}{\partial X} \quad (\text{no sum}).$$

Hence

$$\begin{aligned}
\frac{\partial u_r}{\partial T^{n,\alpha}} &= \hat{t}_1^2 \hat{\eta}^{rr} \frac{\partial}{\partial u_r} \left(\pm \frac{1}{\hat{t}_1} \hat{h}_{\tilde{n}-1, \tilde{\alpha}} \right) \frac{\partial u_r}{\partial X}, \\
&= \hat{t}_1^2 \left(\pm \frac{1}{\hat{t}_1} \hat{\eta}^{rr} \frac{\partial \hat{h}_{\tilde{n}-1, \tilde{\alpha}}}{\partial u_r} \mp \frac{1}{\hat{t}_1^2} \hat{\eta}^{rr} \hat{\eta}_{rr} \hat{h}_{\tilde{n}-1, \tilde{\alpha}} \right), \quad (\text{as } \hat{\eta}_{rr} = \partial_r \hat{t}_1) \\
&= \pm \hat{t}_1 \frac{\partial u_r}{\partial \hat{T}^{\tilde{n}, \tilde{\alpha}}} \mp \hat{h}_{\tilde{n}-1, \tilde{\alpha}} \frac{\partial u_r}{\partial X}.
\end{aligned}$$

□

The canonical coordinates have a natural interpretation in the context of hydrodynamic systems: they are a specific example of a system of *Riemann invariants* for the principal hierarchy. Transformations of this form between equations of hydrodynamic type are known as *reciprocal transformations*. They were originally introduced by Rogers [52]. Ferapontov and Pavlov [27] studied how they act on the Hamiltonian operators of hydrodynamic type.

Theorem 6. *The group $SL(2, \mathbb{C})$ acts on the principal hierarchy of a modular Frobenius manifold by*

$$\frac{\partial}{\partial T^{\alpha, p}} = \pm (ct^N + d) \frac{\partial}{\partial T^{\tilde{\alpha}, \tilde{p}}} \mp h_{\tilde{\alpha}, \tilde{p}-1} \frac{\partial}{\partial X}, \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}),$$

where \pm and $(\tilde{n}, \tilde{\alpha})$ are as above.

Proof. Owing to Proposition 4, one can identify objects with and without hats on a modular Frobenius manifold. Combining this with the results of the above Proposition 16 we have, in particular,

$$\frac{\partial}{\partial T^{\alpha, p}} = \frac{\partial}{\partial \hat{T}^{\alpha, p}}.$$

□

Let us now turn our attention to a couple of examples. The first will be of two distinct two dimensional Frobenius manifolds in dimension two, linked under the inversion symmetry. The second will be of the three dimensional modular Frobenius manifold presented above.

4.2.1 Dispersionless Limits of the David Benney and Harry Dym Hierarchies

The dispersionless limit of the Benney hierarchy may be defined by the dispersionless Lax equations

$$\frac{\partial L}{\partial T_n} = \{ L_{\geq 1}^n, L \}$$

for the Laurent series

$$L = p + v^1(x) + v^2(x)p^{-1},$$

and the canonical Poisson bracket on \mathbb{R}^2 :

$$\{ \cdot, \cdot \} = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial p}.$$

The subscript ≥ 1 denotes the projection onto part of the Laurent series containing strictly positive powers of p . The time T^1 is identified with the spatial variable X . Computing the first couple of flows of the hierarchy, one finds

$$\begin{aligned} \frac{\partial}{\partial T^2} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} &= \frac{\partial}{\partial X} \begin{pmatrix} (v^1)^2 + 2v^2 \\ 2v^1v^2 \end{pmatrix} \\ \frac{\partial}{\partial T^3} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} &= \frac{\partial}{\partial X} \begin{pmatrix} 6v^1v^2 + 6(v^1)^2 \\ 3(v^1)^2v^2 + 3(v^2)^2 \end{pmatrix} \end{aligned}$$

The T^2 flow is called the Benney equation after David Benney. It can be used to model the propagation of water waves.

The hierarchy also admits a Bi-Hamiltonian representation,

$$\frac{\partial v^\sigma}{\partial T^{n,\alpha}} = P_1^{\sigma\varepsilon} \frac{\partial h_{n+1,\alpha}}{\partial v^\varepsilon} = P_2^{\sigma\varepsilon} \frac{\partial h_{n,\alpha}}{\partial v^\varepsilon}, \quad \alpha, \sigma = 1, 2;$$

where

$$P_1 = \begin{pmatrix} 0 & \partial_X \\ \partial_X & 0 \end{pmatrix}; \quad P_2 = \begin{pmatrix} 2\partial_X & v_X^1 \\ v^1\partial_X & v^2\partial_X + v_X^2 \end{pmatrix},$$

and the Hamiltonian densities have the form [65]

$$h_{n,2} = \frac{1}{n!} [\text{Coeff}_{p^{-1}}(L^n)], \quad (4.2.9)$$

$$h_{n,1} = \frac{2}{n!} [\text{Coeff}_{p^{-1}}(L^n(\log L - H_n))], \quad (4.2.10)$$

where as in Chapter 1, H_n denotes the n^{th} harmonic number. In the above flows, $T^n := T^{n,2}$. This is exactly the Bi-Hamiltonian structure generated by the pencil of flat metrics

corresponding to the two dimensional Frobenius manifold

$$F = \frac{1}{2}t_1^2 t_2 + t_2^2 \log t_2; \quad E = t_1 \frac{\partial}{\partial t_1} - 2t_2 \frac{\partial}{\partial t_2}.$$

Indeed,

$$P_1^{\alpha\beta} = \eta^{\alpha\beta} \partial_X; \quad P_2^{\alpha\beta} = g^{\alpha\beta}(t) \partial_X + \Gamma_\gamma^{\alpha\beta} t_X^\gamma.$$

Further, expanding the expressions for the Hamiltonians given by the Lax representation (4.2.9) (4.2.10), we have

$$\begin{aligned} h_{0,2} &= v^1, & h_{0,1} &= v^2, \\ h_{1,2} &= \frac{1}{2}(v^1)^2 + v^2(\log v^2 - 1), & h_{1,1} &= v^1 v^2, \\ h_{2,2} &= \frac{1}{6}(v^1)^3 + v^1 v^2(\log v^2 - 1), & h_{2,1} &= \frac{1}{2}(v^1)^2 v^2 + \frac{1}{2}(v^2)^2, \\ h_{3,2} &= \frac{1}{24}(v^1)^3 + \frac{1}{2}(v^1)^2 v^2(\log v^2 - 1) + \frac{1}{2}(v^2)^2(\log v^2 - \frac{5}{2}), & h_{3,1} &= \frac{1}{6}(v^1)^3 v^2 + \frac{1}{2}v^1(v^2)^2, \end{aligned} \quad (4.2.11)$$

which coincide with the coefficients in the expansions of the deformed flat coordinates (1.3.15), (1.3.16) for this Frobenius manifold.

In Example 14 we computed that the inversion symmetry mapped this Frobenius manifold to another two dimensional Frobenius manifold described by the free energy and Euler field

$$\hat{F} = \frac{1}{2}\hat{t}_1^2 \hat{t}_2 - \log \hat{t}_2; \quad \hat{E} = \hat{t}_1 \frac{\partial}{\partial \hat{t}_1} - 2\hat{t}_2 \frac{\partial}{\partial \hat{t}_2}.$$

We list here some of the coefficients appearing in the expansions of the deformed flat coordinates for this manifold,

$$\begin{aligned} \hat{h}_{0,2} &= \hat{t}^1, & \hat{h}_{0,1} &= \hat{t}^2, \\ \hat{h}_{1,2} &= \frac{1}{2}\hat{t}_1^2 - \frac{1}{2\hat{t}^2}, & \hat{h}_{1,1} &= \hat{t}^1 \hat{t}^2, \\ \hat{h}_{2,2} &= \frac{1}{6}(\hat{t}^1)^3 - \frac{\hat{t}^1}{2\hat{t}^2}, & \hat{h}_{2,1} &= \frac{1}{2}(\hat{t}^1)^2 \hat{t}^2 + \log \hat{t}^2, \\ \hat{h}_{3,2} &= \frac{1}{24}(\hat{t}^1)^4 - \frac{4(\hat{t}^1)^4}{\hat{t}^2} + \frac{1}{12\hat{t}^2}, & \hat{h}_{3,1} &= \frac{1}{6}(\hat{t}^1)^3 \hat{t}^2 + \hat{t}^1(\log \hat{t}^2 + 1), \\ \hat{h}_{4,2} &= \frac{1}{120}(\hat{t}^1)^5 - \frac{(\hat{t}^1)^3}{12\hat{t}^2} + \frac{\hat{t}^1}{12(\hat{t}^2)^2}, & \hat{h}_{4,1} &= \frac{1}{24}(\hat{t}^1)^4 \hat{t}^2 + \frac{1}{2}(\hat{t}^1)^2(\log \hat{t}^2 + 1) - \frac{1}{2\hat{t}^2}(\log \hat{t}^2 + \frac{5}{2}). \end{aligned} \quad (4.2.12)$$

The principal hierarchy of this Frobenius manifold is linked to the dispersionless limit of the Dym hierarchy [65], after Harry Dym. The pencil of metrics for this manifold gives rise to the compatible pair of Hamiltonian operators

$$P_1 = \begin{pmatrix} 0 & \partial_X \\ \partial_X & 0 \end{pmatrix}; \quad P_2 = \begin{pmatrix} \frac{2}{(\hat{t}^2)^2} \partial_X - \frac{2}{(\hat{t}^2)^3} \hat{t}_X^2 & \hat{t}^1 \partial_X - \hat{t}_X^1 \\ \hat{t}^1 \partial_X + 2\hat{t}_X^1 & -2\hat{t}^2 \partial_X - \hat{t}_X^2 \end{pmatrix},$$

which together with the functions $\hat{h}_{n,\alpha}$ can be used to generate the principal hierarchy.

For example, the $\hat{T}^{1,2}$ -flow reads

$$\frac{\partial}{\partial \hat{T}^{1,2}} \begin{pmatrix} \hat{t}^1 \\ \hat{t}^2 \end{pmatrix} = \frac{\partial}{\partial X} \begin{pmatrix} \frac{1}{2(\hat{t}^2)^2} \\ \hat{t}^1 \end{pmatrix}. \quad (4.2.13)$$

Comparing the lists (4.2.11) and (4.2.12), we observe Proposition 15 for this example, for $n \leq 4$:

$$h_{n,2} \left(\hat{t}^1, -\frac{1}{\hat{t}^2} \right) = \frac{1}{\hat{t}^2} \hat{h}_{n+1,1}(\hat{t}^1, \hat{t}^2), \quad h_{n,1} \left(\hat{t}^1, -\frac{1}{\hat{t}^2} \right) = -\frac{1}{\hat{t}^2} \hat{h}_{n-1,2}(\hat{t}^1, \hat{t}^2), \quad \text{for } n = 1, 2, 3, 4.$$

Taking the Benney equation, (recall that $T^2 = T^{2,1}$, i.e. is generated by the Hamiltonian

$$H_{2,1} = \int_{S^1} \left(\frac{1}{2} (v^1)^2 v^2 + \frac{1}{2} (v^2)^2 \right) dX$$

with respect to the operator P_1), and applying the inversion symmetry, we have

$$\begin{aligned} \frac{\partial}{\partial T^{2,1}} \begin{pmatrix} \hat{t}^1 \\ -\frac{1}{\hat{t}^2} \end{pmatrix} &= \frac{\partial}{\partial X} \begin{pmatrix} \frac{1}{2} (\hat{t}^1)^2 - \frac{1}{\hat{t}^2} \\ -\frac{\hat{t}^1}{\hat{t}^2} \end{pmatrix} \\ \Rightarrow \frac{\partial}{\partial T^{2,1}} \begin{pmatrix} \hat{t}^1 \\ \hat{t}^2 \end{pmatrix} &= \begin{pmatrix} \hat{t}^1 \hat{t}_X^1 + \frac{\hat{t}_X^2}{(\hat{t}^2)^2} \\ -\hat{t}_X^1 \hat{t}^2 + \hat{t}^1 \hat{t}_X^2 \end{pmatrix} = -\hat{t}^2 \frac{\partial}{\partial X} \begin{pmatrix} \frac{1}{2(\hat{t}^2)^2} \\ \hat{t}^1 \end{pmatrix} + \hat{t}^1 \frac{\partial}{\partial X} \begin{pmatrix} \hat{t}^1 \\ \hat{t}^2 \end{pmatrix} \\ &= \left(-\hat{t}^2 \frac{\partial}{\partial \hat{T}^{1,2}} + \hat{h}_{0,2} \frac{\partial}{\partial X} \right) \begin{pmatrix} \hat{t}^1 \\ \hat{t}^2 \end{pmatrix}. \end{aligned}$$

This is an example of Proposition 16. In this case the inversion symmetry lifts to a reciprocal transformations between the dispersionless Dym and Benney hierarchies. Of course, one can complete the calculations and verify the results of the proposition for any flow of the principal hierarchies.

4.2.2 A Modular Integrable System

Consider the Frobenius manifold defined by the free energy

$$F = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 - \frac{t_2^4}{16} \gamma(t_3); \quad E = t_1 \frac{\partial}{\partial t_1} + \frac{1}{2} t_2 \frac{\partial}{\partial t_2}, \quad (4.2.14)$$

where γ is some unknown 1-periodic function. For the duration of this example all coordinates will be written with lowered indices ($t_\alpha = t^\alpha$). In order for F to satisfy WDVV, γ must satisfy Chazy's equation,

$$\gamma'''(t_3) = 6\gamma(t_3)\gamma''(t_3) - 9(\gamma'(t_3))^2.$$

We saw in Chapter 2 that this solution to WDVV defined a modular Frobenius manifold.

Recall the diagram depicting how the densities are mapped under the inversion symmetry, outlined in the motivating Example 29 at the beginning of this chapter. Let us compute how this symmetry lifts to the flows.

The flow corresponding to the density $h_{(0,2)}$ is

$$\frac{\partial}{\partial T^{1,2}} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{4}t_2^4\gamma'(t_3) & -\frac{3}{4}t_2^3\gamma''(t_3) \\ 1 & -\frac{3}{2}\gamma(t_3)t_2 & -\frac{3}{4}t_2^2\gamma'(t_3) \\ 0 & 1 & 0 \end{pmatrix} \frac{\partial}{\partial X} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix},$$

Inverting this one finds

$$\begin{aligned} \frac{\partial}{\partial T^{1,2}} \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix} &= \begin{pmatrix} \hat{t}_2 & \frac{3}{4}\hat{t}_2^2\hat{t}_3\gamma'(\hat{t}_3) & \frac{1}{4}\hat{t}_2^3\hat{t}_3\gamma''(\hat{t}_3) \\ -\hat{t}_3 & \hat{t}_2 + \frac{3}{2}\gamma(\hat{t}_3)\hat{t}_2\hat{t}_3 & \frac{3}{4}\hat{t}_2^2\hat{t}_3\gamma'(\hat{t}_3) \\ 0 & -\hat{t}_3 & \hat{t}_2 \end{pmatrix} \frac{\partial}{\partial X} \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix} \\ &= \begin{pmatrix} -\hat{t}_3 \begin{pmatrix} 0 & -\frac{3}{4}\hat{t}_2^4\gamma'(\hat{t}_3) & -\frac{3}{4}\hat{t}_2^3\gamma''(\hat{t}_3) \\ 1 & -\frac{3}{2}\gamma(\hat{t}_3)\hat{t}_2 & -\frac{3}{4}\hat{t}_2^2\gamma'(\hat{t}_3) \\ 0 & 1 & 0 \end{pmatrix} + \hat{t}_2 \mathbf{1} \end{pmatrix} \frac{\partial}{\partial X} \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix} \\ &= (-\hat{t}_3 \hat{M}_{1,2}(\hat{\mathbf{t}}) + h_{(0,2)}(\hat{\mathbf{t}}) \mathbf{1}) \frac{\partial}{\partial X} \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix}, \end{aligned}$$

as predicted by (4.2.8).

Similarly, one can see how the flows corresponding to $\alpha = 1$ and $\alpha = 3$ are related. The flow corresponding to $h_{(1,1)}$ is

$$\frac{\partial}{\partial T^{2,1}} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} t_1 & -\frac{3}{4}t_2^3\gamma'(t_3) - \frac{1}{4}t_2^3t_3\gamma''(t_3) & -\frac{1}{4}t_2^4\gamma''(t_3) - \frac{1}{16}t_2^4t_3\gamma'''(t_3) \\ t_2 & t_1 - \frac{3}{2}\gamma(t_3)t_2^2 - \frac{3}{4}t_2^2t_3\gamma'(t_3) & -\frac{3}{4}t_2^3\gamma'(t_3) - \frac{1}{4}t_2^3t_3\gamma''(t_3) \\ t_3 & t_2 & t_1 \end{pmatrix} \frac{\partial}{\partial X} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}.$$

This may be inverted to give

$$\begin{aligned}
\frac{\partial}{\partial T^{2,1}} \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix} &= \begin{pmatrix} \hat{t}_1 & \frac{1}{4}\hat{t}_2^3\hat{t}_3\gamma''(\hat{t}_3) & \frac{1}{16}\hat{t}_2^4\hat{t}_3\gamma'''(\hat{t}_3) \\ 0 & \hat{t}_1 + \frac{3}{4}\hat{t}_2^2\hat{t}_3\gamma'(\hat{t}_3) & \frac{1}{4}\hat{t}_2^3\hat{t}_3\gamma''(\hat{t}_3) \\ -\hat{t}_3 & 0 & \hat{t}_1 \end{pmatrix} \frac{\partial}{\partial X} \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\frac{1}{4}\hat{t}_2^3\hat{t}_2\gamma''(\hat{t}_3) & -\frac{1}{16}\hat{t}_2^4\gamma'''(\hat{t}_3) \\ -\hat{t}_3 & 0 & -\frac{3}{4}\hat{t}_2^2\gamma'(\hat{t}_3) \\ 1 & 0 & 0 \end{pmatrix} + \hat{t}_1 \mathbf{1} \frac{\partial}{\partial X} \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix} \\
&= (-\hat{t}_3 \hat{M}_{(1,3)}(\hat{t}) + \hat{h}_{(0,3)}(\hat{t}) \mathbf{1}) \frac{\partial}{\partial X} \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{pmatrix}.
\end{aligned}$$

Note the use, in the above example, of the modular transformation properties of solutions of Chazy's equation and their derivatives.

4.3 Almost Duality of Frobenius Manifolds

We saw that the structure of a Frobenius manifold is equivalent to a particular ansatz for a solution to the WDVV equations. As one might expect then, it is possible to relax some of the axioms of a Frobenius manifold and still obtain a solution to the WDVV equations. For example, what happens if one does not stipulate the unity field on the manifold be covariantly constant? We can still obtain a solution to the WDVV equations, and in fact the structure on the tangent spaces is still that of a Frobenius algebra. A good example of a family of solutions for which the unity field is not covariantly constant was found by Feigin & Veselov [28],

$$F^* = \sum_{\alpha \in \mathcal{R}_W} h_\alpha(\alpha, \mathbf{z})^2 \log(\alpha, \mathbf{z})^2. \quad (4.3.1)$$

Here $\mathbf{z} = (z^1, \dots, z^N)$ are a system of flat coordinates for the metric (\cdot, \cdot) , with respect to which the function F^* satisfies the WDVV equations. \mathcal{R}_W denotes the root system corresponding to a finite irreducible Coxeter group W and h_α are a set of constants.

This raises the question: what are the geometric structures defined by these solutions to WDVV that just fail to be Frobenius manifolds?

4.3.1 The Existence of F^* and Twisted Periods

Given a Frobenius manifold, we may define a new multiplication \star by taking the original one and twisting by the Euler field:

$$X \star Y = E^{-1} \circ X \circ Y, \quad \text{for } X, Y \in \Gamma(T\mathcal{M}, \mathcal{M}). \quad (4.3.2)$$

The vector field E^{-1} is the solution of the linear (in the components of E^{-1}) system

$$E^{-1} \circ E = e.$$

So (4.3.2) is well defined whenever E^{-1} is. Let $\mathcal{M}^* = \mathcal{M} \setminus \{t \in \mathcal{M} : E \text{ is not invertible}\}$. Clearly \star is commutative, and has unity field E :

$$E \star X = E^{-1} \circ E \circ X = e \circ X = X.$$

What is more, this multiplication, together with the intersection form (\cdot, \cdot) endow $T^*\mathcal{M}$ with the structure of a Frobenius algebra:

$$g(\omega_1 \star \omega_2, \omega_3) = \iota_E(\omega_1 \star \omega_2 \circ \omega_3) = \iota_E(E^{-1} \circ \omega_1 \circ \omega_2 \circ \omega_3), \quad \forall \omega_1, \omega_2, \omega_3 \in \Gamma(T^*\mathcal{M}, \mathcal{M})$$

is symmetric in ω_1, ω_2 , and ω_3 . Thus we have gone some way to showing that the data $(\mathcal{M}^*, g(\cdot, \cdot), \star, E)$ satisfy the axioms of a Frobenius manifold. It is not difficult to show

the potentiality, i.e. that the 4-tensor ${}^s\nabla_W g(X \star Y, Z)$ is totally symmetric. We refer the reader to [20] for details. Thus, we arrive at

Theorem 7. [20] *Given a Frobenius manifold $(\mathcal{M}, \circ, e, E, \eta)$, there exists a function F^* defined on \mathcal{M}^* such that:*

$$\begin{aligned} {}^*c_{ijk} &= g\left(\frac{\partial}{\partial z^i} \star \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right), \\ &= \frac{\partial^3 F^*}{\partial z^i \partial z^j \partial z^k}. \end{aligned}$$

Moreover, the pair (F^*, g) satisfy the WDVV-equations in the flat coordinates $\{z^i \mid i = 1, \dots, N\}$ of the metric g , where g is the intersection form of the underlying Frobenius manifold.

What has not been discussed so far are the homogeneity conditions usually present in the definition of a Frobenius manifold. Also, there is no reason that the unity field (remember that this will be given by the Euler field of the underlying Frobenius manifold) should be covariantly constant with respect to g . From this point the analysis splits, and the technical results presented all depend on whether $d = 1$ or $d \neq 1$.

Lemma 10. [20] *Let $\{z^1, \dots, z^N\}$ be a system of flat coordinates for the intersection form. If $d \neq 1$ then they can be chosen to satisfy the homogeneity condition*

$$\mathcal{L}_E z^\sigma = \frac{1-d}{2} z^\sigma. \quad (4.3.3)$$

If $d = 1$ then the vector field E is covariantly constant.

Proof. That the set of functions $\{z^1, \dots, z^N\}$ are flat coordinates for the intersection form means they satisfy the system of partial differential equations

$$g^{\alpha\epsilon} \frac{\partial^2 z^\sigma}{\partial t^\beta \partial t^\epsilon} + c_\beta^{\alpha\rho}(t) \left(\nabla E - \frac{1-d}{2} \right)_\rho^\epsilon \frac{\partial z^\sigma}{\partial t^\epsilon} = 0, \quad \sigma = 1, \dots, N.$$

Multiplying through by $g_{\alpha\epsilon}$ and contracting with the Euler field we obtain

$$\begin{aligned} E^\epsilon \frac{\partial^2 z^\sigma}{\partial t^\beta \partial t^\epsilon} + E^\epsilon g_{\beta\nu} c_\epsilon^{\nu\rho}(t) \left(\nabla E - \frac{1-d}{2} \right)_\rho^\epsilon \frac{\partial z^\sigma}{\partial t^\epsilon} &= 0, \quad \sigma = 1, \dots, N, \\ \Leftrightarrow E^\epsilon \frac{\partial^2 z^\sigma}{\partial t^\beta \partial t^\epsilon} + \left(\nabla E - \frac{1-d}{2} \right)_\beta^\epsilon \frac{\partial z^\sigma}{\partial t^\epsilon} &= 0, \quad \sigma = 1, \dots, N, \\ \Leftrightarrow d(\iota_E dz^\sigma) &= \frac{1-d}{2} dz^\sigma, \quad \sigma = 1, \dots, N, \\ \Leftrightarrow \iota_E dz^\sigma &= \frac{1-d}{2} z^\sigma + \text{const.}, \quad \sigma = 1, \dots, N. \end{aligned} \quad (4.3.4)$$

So if $d \neq 1$, we can perform a shift in the variable z^σ so that the homogeneity condition (4.3.3) is satisfied.

If $d = 1$, equation (4.3.4) reads

$$d(\iota_E dz^\sigma) = 0, \quad \text{for } \sigma = 1, \dots, N,$$

which means that the entries of the Euler field are constant in the flat coordinates of the intersection form, i.e. E is covariantly constant. \square

Thus we arrive at a fork in the road. Both routes must be explored in turn.

4.3.2 Almost duality for $d \neq 1$

The results appearing in this subsection first appeared in [20], though have been slightly re-cast to suit our needs.

Lemma 11. *If, for $d \neq 1$, the flat coordinates $\{z^1(t), \dots, z^N(t)\}$ of the intersection form are chosen in such a way that*

$$\mathcal{L}_E z = \frac{1-d}{2} z \tag{4.3.5}$$

and

$$g(dz^a, dz^b) = g^{ab},$$

then

$$t_1 = \eta_{1\alpha} t^\alpha = \frac{1-d}{4} g_{ab} z^a z^b. \tag{4.3.6}$$

Proof. By definition of the intersection form,

$$\begin{aligned} ({}^g\nabla t_1)^\varepsilon &= g(dt_1, dt^\varepsilon) \\ &= \eta_{1\sigma} g(dt^\sigma, dt^\varepsilon) \\ &= \eta_{1\sigma} E^\mu c_\mu^{\sigma\varepsilon} \\ &= E^\mu c_{1\mu}^\varepsilon = E^\varepsilon. \end{aligned}$$

Here ${}^g\nabla t_1$ means the gradient of the function t_1 with respect to the intersection form. Thus, we have shown

$${}^g\nabla t_1 = E.$$

Now equation (4.3.5) implies

$$E = \frac{1-d}{2} z^a \frac{\partial}{\partial z^a},$$

and so

$$g^{ab} \frac{\partial t_1}{\partial z^b} = \frac{1-d}{2} z^a \Rightarrow t_1 = \frac{1-d}{4} g_{ab} z^a z^b.$$

\square

Corollary 1. *If $d \neq 1$, the almost dual prepotential F^* satisfies the homogeneity condition*

$$\sum_i z^i \frac{\partial F^*}{\partial z^i} = 2F^* + \frac{1}{1-d} \sum_{i,j} g_{ij} z^i z^j. \quad (4.3.7)$$

Proof. The components of the metric

$$g_{ij} = \left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right)$$

are constant in the flat coordinates $\{z^i : i = 1, \dots, N\}$. The Euler vector field coincides with the unity. Therefore

$$g_{ij} = \left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k} \right) = \left(E^* \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k} \right) = \sum_i E^i \frac{\partial^3 F^*}{\partial z^i \partial z^j \partial z^k} \quad (4.3.8)$$

$$\Leftrightarrow \frac{2}{1-d} g_{jk} = \sum_i z^i \frac{\partial^3 F^*}{\partial z^i \partial z^j \partial z^k}. \quad (4.3.9)$$

Integrating twice, we obtain the equation (4.3.7). \square

Example 30. *Recall that the Free energy and Euler field*

$$F = \frac{1}{2} t_1^2 t_2 + \frac{1}{2} t_2^2 \left(\log t_2 - \frac{3}{2} \right); \quad E = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} \quad (4.3.10)$$

defines a two-dimensional Frobenius manifold. Computations give

$$g = 2 \frac{\partial}{\partial t_1} \otimes \frac{\partial}{\partial t_1} + t_1 \frac{\partial}{\partial t_1} \otimes \frac{\partial}{\partial t_2} + t_1 \frac{\partial}{\partial t_2} \otimes \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \otimes \frac{\partial}{\partial t_2};$$

$$E^{-1} = \frac{1}{\Delta} \left(t_1 \frac{\partial}{\partial t_1} - 2t_2 \frac{\partial}{\partial t_2} \right); \quad \Delta = t_1^2 - 4t_2.$$

Therefore $\mathcal{M}^ = \mathcal{M} \setminus \Delta$. One finds the flat coordinate system for the intersection form to be given by*

$$t_1 = z_1 + z_2, \quad t_2 = z_1 z_2.$$

In this coordinate system the metric takes an anti-diagonal form,

$$g = \frac{\partial}{\partial z_1} \otimes \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2} \otimes \frac{\partial}{\partial z_1}.$$

Further, we have

$$\begin{aligned}\frac{\partial}{\partial z_1} \star \frac{\partial}{\partial z_1} &= \frac{1}{z_1 - z_2} \frac{\partial}{\partial z_1} + \left(\frac{1}{z_1} + \frac{1}{z_2 - z_1} \right) \frac{\partial}{\partial z_2} \\ \frac{\partial}{\partial z_1} \star \frac{\partial}{\partial z_2} &= \frac{1}{z_2 - z_1} \frac{\partial}{\partial z_1} + \frac{1}{z_1 - z_2} \frac{\partial}{\partial z_2} \\ \frac{\partial}{\partial z_2} \star \frac{\partial}{\partial z_2} &= \frac{1}{z_2 - z_1} \frac{\partial}{\partial z_2} + \left(\frac{1}{z_2} + \frac{1}{z_1 - z_2} \right) \frac{\partial}{\partial z_1},\end{aligned}$$

from which it follows that

$$\begin{aligned}F_{111}^* &= \frac{1}{z_1} + \frac{1}{z_2 - z_1}, \\ F_{112}^* &= \frac{1}{z_1 - z_2}, \\ F_{122}^* &= \frac{1}{z_2 - z_1}, \\ F_{222}^* &= \frac{1}{z_2} + \frac{1}{z_1 - z_2}.\end{aligned}$$

Integrating these equations, we find

$$F^* = \frac{1}{2} (z_1^2 \log z_1 - (z_1 - z_2)^2 \log(z_1 - z_2) + z_2^2 \log z_2).$$

We see that this solution to WDVV fits into the class studied by Feigin & Veselov [28]. Here the Coxeter group is generated by the root system B_2 . If $\alpha_1 = (1, 0)$, $\alpha_2 = (0, 1)$ and $\alpha_3 = (-1, 1)$, we have $h_1 = h_2 = 1$, $h_3 = -1$.

Recall that the vanishing of the curvature of the deformed Euclidean connection was equivalent to the WDVV equations. Therefore we can repeat the construction of flat sections for a 1-parameter family of deformed flat connections on the almost dual manifold \mathcal{M}^* . These are by definition independent solutions of the differential equation

$${}^s\nabla(v)d\zeta_i = 0, \quad i = 1, \dots, N, \quad (4.3.11)$$

where

$${}^s\nabla(v)_Y X = {}^s\nabla_Y X - vX \star Y, \quad \forall X, Y \in \Gamma(T\mathcal{M}^*, \mathcal{M}^*). \quad (4.3.12)$$

We will call solutions to the differential equation (4.3.11) *twisted periods* of the almost dual structure. In the flat coordinates of the intersection form, the twisted periods satisfy

$$\frac{\partial^2 \zeta_l(\mathbf{z}; v)}{\partial z^i \partial z^j} = v c_{ij}^k(\mathbf{z}) \frac{\partial \zeta_l(\mathbf{z}; v)}{\partial z^k}.$$

Proposition 17. *For $d \neq 1$, the twisted periods satisfy the homogeneity conditions*

$$\mathcal{L}_E \zeta(\mathbf{z}, v) = \left(\frac{1-d}{2} + v \right) \zeta(\mathbf{z}, v). \quad (4.3.13)$$

Proof. Recall Cartan's formula for the Lie derivative of a one form along a vector field,

$$\mathcal{L}_E \omega = \iota_E d\omega + d(\iota_E \omega).$$

Let $\omega = d\zeta$. Then, using $d^2 = 0$, we have in the flat coordinates of the intersection form,

$$\begin{aligned} \mathcal{L}_E \left(\frac{\partial \zeta}{\partial z^k} \right) &= \partial_k \left(\frac{1-d}{2} z^i \frac{\partial \zeta}{\partial z^i} \right), \\ &= \frac{1-d}{2} z^i \frac{\partial^2 \zeta}{\partial z^i \partial z^k} + \frac{1-d}{2} \frac{\partial \zeta}{\partial z^k}, \\ &= \frac{1-d}{2} v z^i c_{ik}^* \frac{\partial \zeta}{\partial z^j} + \frac{1-d}{2} \frac{\partial \zeta}{\partial z^k}. \end{aligned}$$

We now use the homogeneity of the second derivatives of the almost dual prepotential,

$$z^i c_{ik}^* = g^{js} z^i \frac{\partial^3 F^*}{\partial z^s \partial z^i \partial z^k} = g^{js} \left(2 \frac{g_{sk}}{1-d} \right) = 2 \frac{\delta_k^j}{1-d}.$$

So

$$\mathcal{L}_E d\zeta = \left(v + \frac{1-d}{2} \right) d\zeta.$$

Using $\mathcal{L}_E d\zeta = d(\mathcal{L}_E \zeta)$ the result follows. \square

Thus demanding the homogeneity (4.3.13) fixes a basis of twisted periods $\{\zeta_1, \dots, \zeta_N\}$. We may again seek power series solutions (recall that $\mathcal{L}_E z^i = \frac{1}{2}(1-d)z^i$)

$$\zeta_i(\mathbf{z}; v) = \sum_{n \geq 0} \zeta_{n,i}(\mathbf{z}) v^n.$$

The equation (4.3.11) then gives a recurrence relation for the coefficients $\zeta_{n,i}(z)$. These 'approximations' inherit their own homogeneity properties from the twisted periods:

$$\mathcal{L}_E \zeta_{n,i}(\mathbf{z}) = \frac{1-d}{2} \zeta_{n,i}(\mathbf{z}) + \zeta_{n-1,i}(\mathbf{z}). \quad (4.3.14)$$

Example 31. *For the example given above, with prepotential*

$$F^* = \frac{1}{2} (z_1^2 \log z_1 - (z_1 - z_2)^2 \log(z_1 - z_2) + z_2^2 \log z_2),$$

we may start the recursion with the seed solutions

$$\zeta_{0,1} = z_1 = g_{1a}z^a, \quad \zeta_{0,2} = z_2 = g_{2a}z^a.$$

Solving the recursion, we find

$$\zeta_{1,1} = (z^2 - z^1) \log(z^2 - z^1) + z^1 \log z^1, \quad (4.3.15)$$

$$\zeta_{1,2} = (z^1 - z^2) \log(z^1 - z^2) + z^2 \log z^2. \quad (4.3.16)$$

The complexity of the functions grows extremely quickly: the third order approximations for both ζ_1 and ζ_2 involve dilogarithms. Note the homogeneity properties (4.3.14). Because for this example we have $d = -1$, we can check

$$(E - 1)\zeta_{1,1} = \left(z^1 \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^2} - 1 \right) \zeta_{1,1} = z^2 = \zeta_{0,1} \quad (4.3.17)$$

$$(E - 1)\zeta_{1,2} = \left(z^1 \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^2} - 1 \right) \zeta_{1,2} = z^1 = \zeta_{0,2}. \quad (4.3.18)$$

These results only hold if $d \neq 1$. In particular, they rely on the normalisation (4.3.6).

4.3.3 Almost Duality for $d = 1$

Lemma 12. Suppose $d = 1$. Then t_1 is a flat coordinate for both η and g .

Proof. Recall that the proof of Lemma 11 uses the statement

$${}^g\nabla t_1 = E,$$

which is independent of d . We have also seen in Lemma 10, that for the case $d = 1$, the Euler field is covariantly constant,

$${}^g\nabla E = 0.$$

The composite of these statements is: if $d = 1$, then ${}^g\nabla^2 t_1 = 0$, i.e. t_1 is a flat coordinate for the metric g . \square

Recall that E plays the role of the unity vector field in the dual picture. Thus when $d = 1$ the unity vector field is covariantly constant, so almost dual Frobenius manifolds at $d = 1$ are even closer to Frobenius manifolds than for those with $d \neq 1$. We will return to this point later. Unfortunately, this appears to be at the expense of losing our grading of $T\mathcal{M}$. As we will see however, we do get a new modular invariance of the almost dual prepotential.

Our next goal will be to define a notion of inversion symmetry solutions to the WDVV that are almost dual to a Frobenius manifold and then look at how the action lifts to the

functions $\zeta_{n,i}$. This, in an analogous fashion to the case for Frobenius manifolds, allow us to study the symmetry on the hydrodynamic type systems associated to these solutions of WDVV.

4.4 Inversion Symmetry and Almost Duality

The aim of this section is to construct the symmetry I^* of the WDVV equations that makes the diagram

$$\begin{array}{ccc} F & \xrightarrow{I} & \hat{F} \\ \downarrow & & \downarrow \\ F^* & \xrightarrow{I^*} & \hat{F}^* \end{array} \quad (4.4.1)$$

commute. Recall that for the case of a Frobenius manifold, the inversion symmetry was defined in the flat coordinate systems. Thus, given the above results, any definition of I^* will have to treat the two cases $d = 1$ and $d \neq 1$ separately. We proceed by analysing examples for which the inversion symmetry I on the underlying Frobenius manifold is well understood.

Example 32. *We have already seen that the two dimensional Frobenius manifolds*

$$\left\{ \begin{array}{l} F = \frac{1}{2}t_1^2 t_2 + t_2^2 (\log t_2 - \frac{3}{2}); \\ E = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2} \end{array} \right\} \xrightarrow{I} \left\{ \begin{array}{l} \hat{F} = \frac{1}{2}\hat{t}_1^2 \hat{t}_2 + \hat{t}_2^2 \log \hat{t}_2; \\ \hat{E} = \hat{t}_1 \frac{\partial}{\partial \hat{t}_1} - 2\hat{t}_2 \frac{\partial}{\partial \hat{t}_2} \end{array} \right\}$$

are related by the inversion symmetry (here $d \neq 1$). Further, we can compute the almost dual prepotentials for both manifolds. We demand that our definition of I^ must map between these almost dual structures:*

$$\left\{ \begin{array}{l} F^* = \frac{1}{2}(z_1^2 \log z_1 - (z_1 - z_2)^2 \log(z_1 - z_2) + z_2^2 \log z_2); \\ E = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \end{array} \right\} \xrightarrow{I^*} \left\{ \begin{array}{l} \hat{F}^* = \frac{1}{2}(\hat{z}_1 - \hat{z}_2)^2 (\log \hat{z}_1 + \log \hat{z}_2 - \log(\hat{z}_1 - \hat{z}_2)); \\ E = -z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} \end{array} \right\}$$

On defining

$$\begin{aligned} \hat{z}_i &= \frac{z_i}{t_1}, \quad i = 1, 2, \\ \hat{g}_{ij} &= g_{ij}, \\ \hat{F}^*(\hat{z}) &= \frac{1}{\hat{t}_1^2} F^*(z(\hat{z})), \end{aligned}$$

the diagram (4.4.1) commutes for this example.

This relationship between the flat coordinate systems holds more generally:

Lemma 13. Suppose $d \neq 1$. Let $\{z^i, i = 1, \dots, N\}$ be a system of flat coordinates for the metric g . Then

$$\left\{ \hat{z}^i = \frac{z^i}{t_1}, \quad i = 1, \dots, N \right\} \quad (4.4.2)$$

is a flat coordinate system for \hat{g} .

To prove this we will use the rather elegant

Lemma 14. For a semisimple Frobenius manifold with canonical coordinate system $\{u_1, \dots, u_N\}$, the corresponding almost dual multiplication \star will be semisimple, with canonical coordinates $\{\tau_1, \dots, \tau_N\}$ on \mathcal{M}^\star given by

$$\tau_i = \log u_i. \quad (4.4.3)$$

Proof. Note that in the canonical coordinates,

$$E = \sum_{i=1}^N u_i \frac{\partial}{\partial u_i} \Rightarrow E^{-1} = \sum_{i=1}^N \frac{1}{u_i} \frac{\partial}{\partial u_i}.$$

Also the definition (4.4.3) gives

$$d\tau_i = \frac{1}{u_i} du_i \Rightarrow \frac{\partial}{\partial \tau_i} = u_i \frac{\partial}{\partial u_i}.$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial \tau_i} \star \frac{\partial}{\partial \tau_j} &= E^{-1} \circ \left(u_i \frac{\partial}{\partial u_i} \right) \circ \left(u_j \frac{\partial}{\partial u_j} \right) \\ &= \frac{1}{u_i} \delta_{ij} u_j^2 \frac{\partial}{\partial u_j} \\ &= \delta_{ij} u_i \frac{\partial}{\partial u_i} \\ &= \delta_{ij} \frac{\partial}{\partial \tau_i}. \end{aligned}$$

□

Corollary 2. The intersection form is Egoroff in the coordinates $\{\tau^1, \dots, \tau^N\}$, with Egoroff potential $t_1(\tau)$.

Let us return to the proof of Lemma 13.

Proof. In canonical coordinates the intersection form reads

$$g = \sum_{i=1}^N \frac{1}{u_i} \frac{\partial t_1(u)}{\partial u_i} du_i \otimes du_i = \sum_{i=1}^N \frac{\partial t_1(\tau)}{\partial \tau_i} d\tau_i \otimes d\tau_i$$

(recall the functions $t_1(u)$ is the Egoroff potential for the metric η). Therefore, just as with the original metrics η and $\hat{\eta}$ of the underlying Frobenius manifolds, the intersection forms are also related by a conformal transformation:

$$g = \frac{1}{t_1^2} \hat{g}.$$

Hence one can employ exactly the same formulae used in the proof of Proposition 15 to show that

$${}^s\nabla dz = 0 \quad \Rightarrow \quad {}^{\hat{s}}\nabla d\left(\frac{z}{t_1}\right) = 0.$$

Because one of the forthcoming Theorems uses the same arguments, and they are technically more involved, we will omit the calculations verifying this statement. To see that the definition (4.4.2) does indeed define a coordinate system, one must compute the determinant of the pullback:

$$\left| \left(\frac{\partial \hat{z}^i}{\partial z^j} \right) \right| = -\frac{1}{t_1^{N+1}}$$

which uses the normalisation (4.3.6). Finally, the Gram matrices for g and \hat{g} coincide:

$$\begin{aligned} \hat{t}_1 &= \frac{1-d}{4} \hat{g}_{ij} \hat{z}^i \hat{z}^j = \frac{1-(2-d)}{4} \hat{g}_{ij} \frac{z^i z^j}{t_1^2} \\ \Leftrightarrow \frac{1}{\hat{t}_1} &= \frac{d-1}{4} \hat{g}_{ij} z^i z^j \\ \Leftrightarrow -t_1 &= \frac{d-1}{4} \hat{g}_{ij} z^i z^j. \end{aligned}$$

Comparing coefficients gives $g_{ij} = \hat{g}_{ij}$. □

How are the flat coordinate systems of two intersection forms g and \hat{g} related at $d = 1$?

Theorem 8. Suppose $d = 1$ and $r_N = 0$ and let $\{z^i : i = 1, \dots, N\}$ be a flat coordinate system for g . Let

$$\hat{z}^1 = \frac{1}{2} \frac{z^\sigma z^\sigma}{z^N}, \quad \hat{z}^\alpha = \frac{z^\alpha}{z^N}, \quad (\text{for } \alpha \neq 1, N), \quad \hat{z}^N = -\frac{1}{z^N}, \quad (4.4.4)$$

where $z^N = t_1$. Then $\{\hat{z}^i : i = 1, \dots, N\}$ are a flat coordinate system for \hat{g} .

Proof. From the above lemma t_1 is a flat coordinate for g and hence we choose $z^N = t_1$. With this $E(z^N) = 0$ and since ${}^s\nabla E = 0$, the vector field E must take the form $E = \sum_{i=1}^{N-1} c_i \frac{\partial}{\partial z^i}$ for some constants c_i . Using the freedom to redefine the p^i for $i \neq N$ one may set

$$E = \frac{\partial}{\partial z^1}.$$

With this $g(E, E) = \eta(E^{-1} \circ E, E) = \eta(e, E) = r_N = 0$ (again since $d = 1$). Thus from Lemma

1 [16] one may redefine coordinates so

$$g_{ij} = \delta_{i+j, N+1}$$

in the $\{z^i\}$ -coordinates. The process of proving that the coordinates defined above are indeed flat for \hat{g} we employ the same method as before. Because the calculations for \hat{z}^i , for $i = 2, \dots, N$ are analogous to ones given earlier, we will just give details of the calculations for \hat{z}^1 which is slightly more involved.

To this end, recall relationships (4.2.5), (4.2.6). Let us compute the transformation rules for the coordinate differentials. The canonical coordinates of \circ are fixed by the inversion symmetry, and by virtue of equation (4.4.3), so are those of \star . In the canonical coordinates $\{\tau_1, \dots, \tau_N\}$ of the intersection form \hat{g} we have for (4.2.5) ($i \neq j$):

$${}^{\hat{g}}\nabla_i \frac{\partial z^k}{\partial \tau_j} = {}^g\nabla_i \frac{\partial z^k}{\partial \tau_j} + \partial_i \log t_1 \frac{\partial z^k}{\partial \tau_j} + \partial_j \log t_1 \frac{\partial z^k}{\partial \tau_j}; \quad \partial_i := \frac{\partial}{\partial \tau_i}.$$

If $i = j$,

$$\begin{aligned} {}^{\hat{g}}\nabla_i \frac{\partial z^k}{\partial \tau_i} &= {}^g\nabla_i \frac{\partial z^k}{\partial \tau_i} + \partial_i \log t_1 \frac{\partial z^k}{\partial \tau_i} - \partial_i \log t_1 \sum_{j \neq i} \frac{\partial z^k}{\partial \tau_j} \\ &= {}^g\nabla_i \frac{\partial z^k}{\partial \tau_i} + 2\partial_i \log t_1 \frac{\partial z^k}{\partial \tau_i} - \delta_1^k \partial_i \log t_1, \end{aligned}$$

because $\sum_{i=1}^N \partial_i z^k = E(z^k) = \delta_1^k$, as we chose $E = \partial_{z^1}$. Therefore, using ${}^g\nabla dz^i = 0$, for $i = 1, \dots, N$, we have

$$\begin{aligned} {}^{\hat{g}}\nabla dz^i &= 2d \log t_1 \otimes dz^i - \delta_1^i \sum_{k=1}^N \partial_k \log t_1 d\tau_k \otimes d\tau_k. \\ &= 2d \log t_1 \otimes dz^i - \frac{\delta_1^i}{z^N} g \end{aligned} \tag{4.4.5}$$

because $t_1 = z^N$ is the Egoroff potential for g . We need to show

$${}^{\hat{g}}\nabla d\hat{z}^1 = {}^{\hat{g}}\nabla \left\{ d \left(z^1 + \frac{1}{2z^N} \sum_{i,j \neq 1} g_{ij} z^i z^j \right) \right\} = 0.$$

Un-packing the right hand side of this expression, and using (4.4.5) we have,

$$\begin{aligned}
\hat{g}\nabla d\hat{z}^1 &= \hat{g}\nabla dz^1 + \frac{1}{2z^N} \left\{ \sum_{i=2}^{N-1} (2dz^i \otimes dz_i + z^i g \nabla dz_i + z_i g \nabla dz^i) \right\} \\
&\quad - \frac{1}{2(z^N)^2} \sum_{i=2}^{N-1} (z^i dz_i + z_i dz^i) \otimes dz^N + \frac{1}{(z^N)^3} \sum_{i=2}^{N-1} z_i z^i dz^N \otimes dz^N \\
&\quad - \frac{1}{2(z^N)^2} \left\{ \left(\sum_{i=2}^{N-1} z^i dz_i + z_i dz^i \right) \otimes dz^N - \sum_{i=2}^{N-1} z^i z_i g \nabla dz^N \right\} \\
&= 2d \log t_1 \otimes dz^1 - \frac{1}{z^N} g + \frac{1}{2z^N} \left\{ \sum_{i=2}^{N-1} (2dz^i \otimes dz_i + z^i 2d \log t_1 \otimes dz_i + z_i 2d \log t_1 \otimes dz^i) \right\} \\
&\quad - \frac{1}{2(z^N)^2} \sum_{i=2}^{N-1} (z^i dz_i + z_i dz^i) \otimes dz^N + \frac{1}{(z^N)^3} \sum_{i=2}^{N-1} z_i z^i dz^N \otimes dz^N \\
&\quad - \frac{1}{2(z^N)^2} \left\{ \left(\sum_{i=2}^{N-1} z^i dz_i + z_i dz^i \right) \otimes dz^N - \sum_{i=2}^{N-1} z^i z_i 2d \log t_1 \otimes dz^N \right\} \\
&= \frac{2}{z^N} dz^1 \otimes dz^N - \frac{1}{z^N} g + \frac{2}{z^N} \sum_{i=2}^{N-1} dz_i \otimes dz^i = 0.
\end{aligned}$$

So $\{\hat{z}^1, \dots, \hat{z}^N\}$ are flat functions for \hat{g} . Further they comprise a coordinate system:

$$\left| \left(\frac{\partial \hat{z}^i}{\partial \hat{z}^j} \right) \right| = -\frac{1}{\hat{t}_1^N}.$$

Using the fact that the metrics g and \hat{g} are related by a conformal transformation, we have

$$\hat{g}_{pq} = t_1^{-2} \frac{\partial z^i}{\partial \hat{z}^p} \frac{\partial z^j}{\partial \hat{z}^q} g_{ij}.$$

This yields, using the definition (4.4.4) $g_{ij} = \hat{g}_{ij}$. □

We are now ready to define the dual inversion symmetry I^* in full.

Theorem 9. *Let F define a Frobenius manifold and let \hat{F} denote the induced manifold under the action of the symmetry I . Let F^* and \hat{F}^* denote the corresponding almost dual structures. Then I^* , the induced symmetry acts as:*

- Case I: $d \neq 1$:

$$\begin{aligned}
\hat{z}^i &= \frac{z^i}{t_1}, \quad i = 1, \dots, N, \\
\hat{g}_{ab} &= g_{ab}, \\
\hat{F}^*(\hat{\mathbf{z}}) &= \frac{F^*(\mathbf{z}(\hat{\mathbf{z}}))}{t_1^2}
\end{aligned} \tag{4.4.6}$$

where $t_1 = \frac{(1-d)}{4} g_{ab} z^a z^b$.

- Case II: $d = 1$:

$$\begin{aligned}\hat{z}^1 &= \frac{1}{2} \frac{z_\sigma z^\sigma}{t_1}, \quad \hat{z}^i = \frac{z^i}{t_1}, \quad i = 2, \dots, N-1, \quad \hat{z}^N = -\frac{1}{t_1}, \\ \hat{g}_{ab} &= g_{ab}, \\ \hat{F}^*(\hat{\mathbf{z}}) &= (\hat{z}^N)^2 F(\mathbf{z}(\hat{\mathbf{z}})) + \frac{1}{2} \hat{z}^1 \hat{z}_\sigma \hat{z}^\sigma,\end{aligned}$$

where $t_1 = z^N$.

Note, in both cases z^N is the Egoroff potential for the metric η .

Proof. Recall (4.3.7) that for $d \neq 1$ the dual prepotential satisfies the homogeneity condition

$$\sum_\alpha z^\alpha \frac{\partial F^*}{\partial z^\alpha} = 2F^* + \frac{1}{1-d} g_{\alpha\beta} z^\alpha z^\beta,$$

from which it follows that

$$\sum_\sigma z^\sigma \frac{\partial^2 F^*}{\partial z^\sigma \partial z^\kappa} = \frac{\partial F^*}{\partial z^\kappa} + \frac{2z_\kappa}{1-d}; \quad (4.4.7)$$

$$\sum_\sigma z^\sigma \frac{\partial^3 F^*}{\partial z^\sigma \partial z^\kappa \partial z^\varepsilon} = \frac{\partial^2 F^*}{\partial z^\kappa \partial z^\varepsilon} + \frac{2g_{\kappa\varepsilon}}{1-d}. \quad (4.4.8)$$

Using this and the explicit coordinates given in Proposition 9, equation (4.4.6), one finds that

$$\frac{\partial}{\partial \hat{z}^\alpha} = -t_1 \frac{\partial}{\partial z^\alpha} - z_\alpha E,$$

where E is the Euler field. From this it follows that

$$\begin{aligned}\frac{\partial}{\partial \hat{z}^\alpha} \left(\frac{F^*}{t_1^2} \right) &= -\frac{1}{t_1} \frac{\partial F^*}{\partial z^\alpha} - \frac{2}{t_1(1-d)} z_\alpha, \\ \frac{\partial^2}{\partial \hat{z}^\alpha \partial \hat{z}^\beta} \left(\frac{F^*}{t_1^2} \right) &= \frac{\partial^2 F^*}{\partial z^\alpha \partial z^\beta} - \frac{4g_{\alpha\beta}}{1-d} - \frac{2}{t_1(1-d)} z_\alpha z_\beta, \\ \frac{\partial^3}{\partial \hat{z}^\alpha \partial \hat{z}^\beta \partial \hat{z}^\gamma} \left(\frac{F^*}{t_1^2} \right) &= -t_1 \frac{\partial^3 F^*}{\partial z^\alpha \partial z^\beta \partial z^\gamma} + \frac{2}{1-d} (g_{\alpha\beta} z_\gamma + g_{\alpha\gamma} z_\beta + g_{\beta\gamma} z_\alpha) - \frac{2}{t_1(1-d)} z_\alpha z_\beta z_\gamma.\end{aligned}$$

Inserting this into the WDVV equations we obtain 25 terms in the left and right hand sides which pair off and cancel. So \hat{F}^* satisfies the WDVV equations in the $\{\hat{z}^i\}$ -variables.

If $d = 1$ the proof is identical to the original inversion symmetry as presented in [16]. □

It remains to show how the deformed flat coordinates $\xi(z; v)$ for ${}^s\nabla$ behave under inversion symmetry. This will be done in the next section. We will end this section with a couple of examples.

Example 33 ($d \neq 1$). *Given an irreducible Coxeter group W of rank N , the Saito construction gives a Frobenius manifold structure on the orbit space \mathbb{C}^N/W . The almost dual prepotential takes the form*

$$F^*(\mathbf{z}) = \frac{1}{4} \sum_{\alpha \in R_W} (\alpha, \mathbf{z})^2 \log(\alpha, \mathbf{z})^2$$

where $(\ , \)$ is the metric g .

Application of the I^* transform (recall $d \neq 1$ for these examples) yields the solution

$$\hat{F}^*(\hat{\mathbf{z}}) = \frac{1}{4} \sum_{\alpha \in R_W} (\alpha, \hat{\mathbf{z}})^2 \log(\alpha, \hat{\mathbf{z}})^2 - \frac{h}{4}(\hat{\mathbf{z}}, \hat{\mathbf{z}}) \log(\hat{\mathbf{z}}, \hat{\mathbf{z}})$$

where h is defined by the relation $\sum_{\alpha \in R_W} (\alpha, \mathbf{z})^2 = h(\mathbf{z}, \mathbf{z})$ (and hence depends on the normalization of the roots $\alpha \in R_W$).

Thus the original solution is recovered but with the addition of a new radial term. Such solutions have been constructed directly (i.e. without knowledge of its geometric origins) in [40].

Example 34 ($d = 1$). *Given the Weyl groups A_N and B_N with Lie algebra \mathfrak{g} with Cartan subalgebra \mathfrak{h} one may construct the so-called Jacobi group $J(\mathfrak{h})$ and orbit space $\Omega/J(\mathfrak{g})$ where $\Omega = \mathbb{C} \oplus \mathfrak{h} \oplus \mathbb{H}$ [6]. This orbit space carries the structure of a Frobenius manifold and it was shown by Riley and Strachan [51] that the dual prepotential takes the form*

$$F^*(u, \mathbf{z}, \tau) = \frac{1}{2} \tau u^2 - \frac{1}{2} u(\mathbf{z}, \mathbf{z}) + \sum_{\alpha \in \mathfrak{U}} h(\alpha, \mathbf{z}, \tau)$$

Here the function h is essentially the elliptic trilogarithm introduced by Beilinson and Levin [5] and the set \mathfrak{U} contains certain vectors - an elliptic generalization of classical root systems. The basic function h satisfies the modularity property (c.f. Example 5)

$$h\left(\frac{z}{\tau}\right) = \frac{1}{\tau^2} h(z, \tau) - \frac{z^4}{4! \tau^3}$$

up to quadratic terms. The proof that this almost dual prepotential lies at a fixed point of the inversion symmetry may be found in [61].

4.5 The Inversion Symmetry and Twisted Periods

With a definition of the inversion symmetry on the space of almost dual Frobenius manifolds in place, we can now study how the symmetry lifts to the corresponding twisted periods and dispersionless principal hierarchies associated to the solutions of WDVV.

Proposition 18. *Suppose $d \neq 1$. Let $\{\zeta_i(\mathbf{z}; \nu), i = 1, \dots, N\}$ and $\{\hat{\zeta}_i(\hat{\mathbf{z}}; \nu), i = 1, \dots, N\}$ be fundamental sets of twisted periods for two solutions F^* and \hat{F}^* of the WDVV equations that are linked by the almost dual inversion symmetry of Theorem 9. Then if*

$$\zeta_i(\mathbf{z}; \nu) = \sum_{n \geq 0} \zeta_{n,i}(\mathbf{z}) \nu^n; \quad \hat{\zeta}_i(\hat{\mathbf{z}}; \nu) = \sum_{n \geq 0} \hat{\zeta}_{n,i}(\hat{\mathbf{z}}) \nu^n$$

we have

$$\zeta_{n,i} \left(\frac{\hat{\mathbf{z}}}{\hat{t}_1} \right) = \frac{1}{\hat{t}_1} \hat{\zeta}_{n,i}(\hat{\mathbf{z}}).$$

Proof. Again, we first need to show that the functional form of the proposition is correct. This is the same as in the proof of Proposition 15. In the canonical coordinates $\{\tau_i, i = 1, \dots, N\}$ the recursion relation becomes

$${}^s\nabla_i \frac{\partial}{\partial \tau_j} \zeta_{n,s}(\mathbf{z}) = \delta_{ij} \frac{\partial}{\partial \tau_j} \zeta_{n-1,s}(\mathbf{z}),$$

and so the same technique applies. Where this proof differs (slightly) is in the identification of the labels. Recall that the basis of solutions $\{\hat{\zeta}_i(\hat{\mathbf{z}}; \nu), i = 1, \dots, N\}$ was fixed by the homogeneity condition

$$\mathcal{L}_E \zeta_i(\mathbf{z}; \nu) = \left(\frac{d-1}{2} + \nu \right) \zeta_i(\mathbf{z}; \nu),$$

which means that the coefficients were fixed by

$$\mathcal{L}_E \zeta_{n,i}(\mathbf{z}) = \frac{1-d}{2} \zeta_{n,i}(\mathbf{z}) + \zeta_{n-1,i}(\mathbf{z}).$$

The dual inversion symmetry fixes the canonical coordinates, so it also fixes the unity $E = \hat{E}$, up to a re-scaling. Therefore

$$\mathcal{L}_{\hat{E}} \zeta_{n,i} \left(\frac{\hat{\mathbf{z}}}{\hat{t}_1} \right) = \mathcal{L}_E \zeta_{n,i}(\mathbf{z}) = \left(\frac{1-d}{2} \right) \zeta_{n,i}(\mathbf{z}) + \zeta_{n-1,i}(\mathbf{z}).$$

On the other hand,

$$\begin{aligned}
\mathcal{L}_{\hat{E}} \left(\frac{1}{\hat{t}_1} \hat{\zeta}_{n,i}(\hat{\mathbf{z}}) \right) &= \frac{1}{\hat{t}_1} \mathcal{L}_{\hat{E}} \hat{\zeta}_{n,i}(\hat{\mathbf{z}}) - \frac{1}{\hat{t}_1^2} \hat{\zeta}_{n,i}(\hat{\mathbf{z}}) \mathcal{L}_{\hat{E}} \hat{t}_1 \\
&= \frac{1}{\hat{t}_1} \left(\frac{1-\hat{d}}{2} \hat{\zeta}_{n,i}(\hat{\mathbf{z}}) + \hat{\zeta}_{n-1,i}(\hat{\mathbf{z}}) \right) - \frac{1}{\hat{t}_1^2} \hat{\zeta}_{n,i}(\hat{\mathbf{z}}) (1-\hat{d}) \hat{t}_1 \\
&= \frac{1}{\hat{t}_1} \left(\frac{1-(2-d)}{2} \hat{\zeta}_{n,i}(\hat{\mathbf{z}}) \right) + \frac{1}{\hat{t}_1} \hat{\zeta}_{n-1,i}(\hat{\mathbf{z}}) - \frac{1}{\hat{t}_1^2} (1-(2-d)) \hat{\zeta}_{n,i}(\hat{\mathbf{z}}) \hat{t}_1 \\
&= \left(\frac{1-d}{2} \right) \frac{1}{\hat{t}_1} \hat{\zeta}_{n,i}(\hat{\mathbf{z}}) + \frac{1}{\hat{t}_1} \hat{\zeta}_{n-1,i}(\hat{\mathbf{z}}).
\end{aligned}$$

So the normalisation conditions are satisfied. \square

Of course, the flat pencil of metrics that allowed us to construct the principal hierarchy still exists, only now we obtain a hydrodynamic type system on the flat coordinates of the intersection form. The almost dual principal hierarchy reads

$$\frac{\partial z^\sigma}{\partial \mathcal{T}^{k,\beta}} = \left\{ \int_{S^1} z^\sigma(X) dX, \int_{S^1} \zeta_{k+1,\beta} dX \right\}_2 = \left\{ \int_{S^1} z^\sigma(X) dX, \int_{S^1} \zeta_{k,\beta} dX \right\}_1, \quad k \geq 0, \beta = 1, \dots, N,$$

where the Poisson brackets are as for the principal hierarchy of the underlying Frobenius manifold. The spatial derivative operator is as for the principal hierarchy,

$$\frac{\partial}{\partial X} := \left\{ \cdot, \int_{S^1} t^N(X) dX \right\}_2.$$

Note that if $d \neq 1$ the induced map between the principal hierarchies arising from these almost dual solutions is different to that for Frobenius manifolds because the twisted periods behave differently to the deformed flat coordinates of the underlying Frobenius manifold.

Corollary 3. *For $d \neq 1$, the dual inversion symmetry I^* acts on the principal hierarchy by*

$$\frac{\partial}{\partial \mathcal{T}^{n,i}} = \hat{t}_1 \frac{\partial}{\partial \hat{\mathcal{T}}^{n,i}} - \hat{\zeta}_{n-1,i}(\mathbf{z}) \frac{\partial}{\partial X}. \quad (4.5.1)$$

Example 35. *We saw in example 4.2.1, that the inversion symmetry led to a reciprocal transformation*

$$\text{Dispersionless Benney Hierarchy} \xrightarrow{\mathcal{R}} \text{Dispersionless Dym Hierarchy}$$

of the principal hierarchies corresponding to the Frobenius manifolds (1.2.14), (1.2.15) respectively. Example 32 showed that the almost dual structures constructed from these manifolds were linked by the almost dual inversion symmetry for $d \neq 1$ (4.4.6). Therefore, these almost dual structures provide an explicit example of Corollary 3.

The first couple of flows corresponding to the almost dual solution

$$F^* = \frac{1}{2} (z_1^2 \log z_1 - (z_1 - z_2)^2 \log(z_1 - z_2) + z_2^2 \log z_2); \quad E = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$$

read

$$\begin{aligned} \frac{\partial}{\partial \mathcal{T}^{1,1}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} \frac{1}{z_1 - z_2} & -\frac{1}{z_1 - z_2} \\ \frac{1}{z_1} - \frac{1}{z_1 - z_2} & \frac{1}{z_1 - z_2} \end{pmatrix} \frac{\partial}{\partial X} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \\ \frac{\partial}{\partial \mathcal{T}^{1,2}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{z_1 - z_2} & \frac{1}{z_2} + \frac{1}{z_1 - z_2} \\ \frac{1}{z_1 - z_2} & -\frac{1}{z_1 - z_2} \end{pmatrix} \frac{\partial}{\partial X} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \end{aligned}$$

Those for the solution

$$\hat{F}^* = \frac{1}{2} (\hat{z}_1 - \hat{z}_2)^2 (\log \hat{z}_1 + \log \hat{z}_2 - \log(\hat{z}_1 - \hat{z}_2)); \quad E = -z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}$$

read

$$\begin{aligned} \frac{\partial}{\partial \hat{\mathcal{T}}^{1,1}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} \frac{1}{\hat{z}_1 - \hat{z}_2} - \frac{1}{\hat{z}_1} & -\frac{1}{\hat{z}_2} - \frac{1}{\hat{z}_1 - \hat{z}_2} \\ \frac{\hat{z}_2}{\hat{z}_1} + \frac{1}{\hat{z}_1} - \frac{1}{\hat{z}_1 - \hat{z}_2} & \frac{1}{\hat{z}_1 - \hat{z}_2} - \frac{1}{\hat{z}_1} \end{pmatrix} \frac{\partial}{\partial X} \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix}, \\ \frac{\partial}{\partial \hat{\mathcal{T}}^{1,2}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{\hat{z}_2} - \frac{1}{\hat{z}_1 - \hat{z}_2} & \frac{\hat{z}_1}{\hat{z}_2} + \frac{1}{\hat{z}_1 - \hat{z}_2} + \frac{1}{\hat{z}_2} \\ \frac{1}{\hat{z}_1 - \hat{z}_2} - \frac{1}{\hat{z}_1} & -\frac{1}{\hat{z}_2} - \frac{1}{\hat{z}_1 - \hat{z}_2} \end{pmatrix} \frac{\partial}{\partial X} \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix}. \end{aligned}$$

Using the normalisation $t^N = \frac{1-d}{2} \sum_{a,b} g_{ab} z_a z_b$, it is a straightforward exercise to observe the symmetry (4.5.1) between these flows.

As we have seen, the study of almost dual solutions to the WDVV equations makes a clear distinction between the two cases $d = 1$ and $d \neq 1$. In [20], Dubrovin gives a recipe for how to reconstruct a Frobenius manifold from an almost dual solution to the WDVV equations if $d \neq 1$. The proof uses, in particular, the homogeneity properties (4.3.13) of the twisted periods (see [20], Lemma 3.12, Lemma 3.13). As we have seen, these are absent when $d = 1$, and the reconstruction problem is open. Instead, we have an invariance of F^* under the inversion symmetry and covariant *constancy* of the unity field E . It would be interesting to see if one could use these properties, specific to $d = 1$, to give a reconstruction theorem in this case.

From the point of view of bi-Hamiltonian systems, it is an interesting to study how equations of hydrodynamic type arising from pencils of metrics in the same conformal class are related. The inversion symmetry of Frobenius manifolds is an example of such a conformal transformation. It turns out that in some sense it is always the case that metrics in the same conformal class give hydrodynamic systems related by a conformal transformation. See the paper [27] of Ferapontov and Pavlov and the references therein.

Chapter 5

The Inversion Symmetry at Genus One

5.1 Introduction

We have seen in earlier chapters how to construct from an arbitrary semi-simple Frobenius manifold an integrable hierarchy of hydrodynamic type PDEs called the *principal hierarchy*. We have also seen how hydrodynamic type equations arise as the dispersionless limits of integrable equations of higher order. For example, we saw how Burger's equation arose as the dispersionless limit of the KdV equation. A natural question to ask therefore is can one do the opposite: given a hydrodynamic type PDE, can one successfully reconstruct the dispersive terms whilst retaining integrability?

From the point of view of the bi-Hamiltonian structure of the evolution equations, obtaining the dispersive hierarchies from their dispersionless limits must be done by obtaining deformations of the various objects involved in defining the equations of hydrodynamic type: both the Poisson brackets and the Hamiltonians defining the flows. For an arbitrary semi-simple Frobenius manifold, this was done to first order, or to *one-loop*, by Dubrovin and Zhang in their 1998 work [23].

Theorem 10. [23] *There exists a unique hierarchy of the form*

$$\frac{\partial \mathbf{t}}{\partial T^{n,\alpha}} = K_{[0];n,\alpha}(\mathbf{t}, \partial_X \mathbf{t}) + \varepsilon^2 [K'(\mathbf{t})_{[1];n,\alpha;\lambda} t_{XXX}^\lambda + K'(\mathbf{t})_{[1];n,\alpha;\lambda\nu} t_{XX}^\lambda t_X^\nu + K'(\mathbf{t})_{[1];n,\alpha;\lambda\nu\mu} t_X^\lambda t_X^\nu t_X^\mu] + \mathcal{O}(\varepsilon^4) \quad (5.1.1)$$

such that the function $\mathbf{t}(T) = (t_1(T), \dots, t_N(T))$ satisfies the underlying principal hierarchy (5.1.1) up to order ε^4 for an arbitrary solution $v(T)$ of (3.2.4),

$$t_\alpha(T) = v_\alpha(T) + \varepsilon^2 \frac{\partial^2}{\partial T^{\alpha,0} \partial T^{1,0}} \left[\frac{1}{24} \log \det c_{\mu\nu\sigma} t_X^\sigma + G(t) \right]_{t=v(T)} + \mathcal{O}(\varepsilon^4). \quad (5.1.2)$$

The functions $K'(\mathbf{t})_{\beta,p;\lambda}$, $K'(\mathbf{t})_{\beta,p;\lambda\nu}$, $K'(\mathbf{t})_{\beta,p;\lambda\nu\mu}$, and $G(\mathbf{t})$ are analytic functions on the Frobenius manifold.

The hierarchy (5.1.1) admits a representation

$$\frac{\partial t}{\partial T^{n,\alpha}} = \{t(X), \mathcal{H}_{n,\alpha} + \varepsilon^2 \delta \mathcal{H}_{n,\alpha}' + \varepsilon^2 \delta \mathcal{H}_{n,\alpha}''\}_{[0]} + \varepsilon^2 \{t(X), \mathcal{H}_{n,\alpha}\}_{[1]}' + \varepsilon^2 \{t(X), \mathcal{H}_{n,\alpha}\}_{[1]}'' + \mathcal{O}(\varepsilon^4) \quad (5.1.3)$$

where

$$\{\mathcal{H}, \mathcal{F}\}_{[0]} = \int_{S^1} \frac{\delta \mathcal{H}}{\delta t^\alpha} P_{[0]}'^{\alpha\beta}(\mathbf{t}) \frac{\delta \mathcal{F}}{\delta t^\beta} dX, \quad (5.1.4)$$

$$\{\mathcal{H}, \mathcal{F}\}_{[1]}' = \int_{S^1} \frac{\delta \mathcal{H}}{\delta t^\sigma} P_{[1]}'^{\alpha\beta}(\mathbf{t}) \frac{\delta \mathcal{F}}{\delta t^\sigma} dX, \quad (5.1.5)$$

$$\{\mathcal{H}, \mathcal{F}\}_{[1]}'' = \int_{S^1} \frac{\delta \mathcal{H}}{\delta t^\alpha} P_{[1]}''^{\alpha\beta}(\mathbf{t}) \frac{\delta \mathcal{F}}{\delta t^\beta} dX \quad (5.1.6)$$

for

$$\begin{aligned}
P_{[0]'}^{\alpha\beta}(\mathbf{t}) &= \eta^{\alpha\beta} \partial_X \\
P_{[1]'}^{\alpha\beta}(\mathbf{t}) &= \frac{1}{24} \left(f^{\alpha\beta}(\mathbf{t}) \partial_X^3 + \frac{3}{2} \partial_X (f^{\alpha\beta}(\mathbf{t})) \partial_X^2 + \frac{1}{2} \partial_X^2 (f^{\alpha\beta}(\mathbf{t})) \partial_X \right) \\
P_{[1]''}^{\alpha\beta}(\mathbf{t}) &= a^{\alpha\beta}(\mathbf{t}) \partial_X^3 + \left(\frac{3}{2} \partial_X (a^{\alpha\beta}(\mathbf{t})) + b^{\alpha\beta}(\mathbf{t}, \partial_X \mathbf{t}_X) \right) \partial_X + (\partial_X b^{\alpha\beta}(\mathbf{t}, \partial_X \mathbf{t}) + \partial_X^2 (a^{\alpha\beta}(\mathbf{t}))) \partial_X
\end{aligned}$$

and

$$\begin{aligned}
f^{\alpha\beta}(\mathbf{t}) &= \eta^{\mu\nu} c_{\mu\nu}^{\alpha\beta} := \eta^{\mu\nu} \eta^{\alpha\sigma} \eta^{\beta\kappa} \frac{\partial^4 F}{\partial t^\sigma \partial t^\kappa \partial t^\mu \partial t^\nu}, \\
a^{\alpha\beta}(\mathbf{t}) &= 2c^{\alpha\beta\nu} \frac{\partial G}{\partial t^\nu}, \\
b^{\alpha\beta}(\mathbf{t}) &= \frac{3}{2} \partial_X a^{\alpha\beta} + \frac{\partial^2 G}{\partial t^\sigma \partial t^\rho} \left(c_\mu^{\alpha\sigma} \eta^{\beta\rho} - c_\mu^{\beta\sigma} \eta^{\alpha\rho} \right) t^{\mu_X}.
\end{aligned}$$

The Hamiltonians of the deformed hierarchy are given by

$$\mathcal{H}_{n,\alpha} = \int_{S^1} h_{n,\alpha}(\mathbf{t}) dX, \quad (5.1.7)$$

$$\delta \mathcal{H}_{n,\alpha}' = \int_{S^1} \Xi_{n+1,\alpha;\sigma\nu}(\mathbf{t}) t_X^\sigma t_X^\nu dX, \quad (5.1.8)$$

$$\delta \mathcal{H}_{n,\alpha}'' = \int_{S^1} \Upsilon_{n+1,\alpha;\sigma\nu}(\mathbf{t}) t_X^\sigma t_X^\nu dX, \quad (5.1.9)$$

where the functions $h_{n,\alpha}(\mathbf{t})$ were defined in (3.2.1), and

$$\begin{aligned}
\Xi_{0,\alpha;\sigma\nu}(\mathbf{t}) &= 0, \\
\Xi_{n+1,\alpha;\sigma\nu}(\mathbf{t}) &= \frac{1}{24} \left(c_{\mu\beta}^{\mu\sigma} c_{\nu\gamma}^{\kappa} - c_\gamma^{\mu\nu} c_{\mu\nu\beta}^{\kappa} \right) \frac{\partial h_{n,\alpha}}{\partial t^\kappa} - \frac{1}{24} c_{\zeta\xi}^\gamma c_\nu^{\zeta\sigma} c_{\sigma\mu}^\zeta \frac{\partial h_{n-1,\alpha}}{\partial t^\gamma}, \\
\Upsilon_{n+1,\alpha;\sigma\nu}(\mathbf{t}) &= c_{\zeta\nu}^\gamma c_\sigma^{\mu\zeta} \frac{\partial h_{n,\alpha}}{\partial t^\gamma} \frac{\partial G}{\partial t^\mu}.
\end{aligned}$$

The Hamiltonians $\mathcal{H}_{n,\alpha} + \varepsilon^2 \delta \mathcal{H}_{n,\alpha}' + \varepsilon^2 \delta \mathcal{H}_{n,\alpha}''$ commute pairwise with respect to the Poisson bracket

$$\{ \cdot, \cdot \}_{[0]} + \varepsilon^2 \{ \cdot, \cdot \}_{[1]'} + \varepsilon^2 \{ \cdot, \cdot \}_{[1]''}$$

modulo $\mathcal{O}(\varepsilon^4)$.

The proof of this theorem is beyond the scope of our present discussions; the reader is referred to [23] for details. Our present discussion will focus on how the inversion symmetry acts on these deformations of the principal hierarchy.

A couple of remarks:

1. The hierarchy (5.1.1) is a deformation of the principal hierarchy in the sense that

its dispersionless limit coincides with the principal hierarchy:

$$K_{[0];n,\alpha}^\sigma(t, t_X) = \eta^{\sigma\nu} c_{\nu\zeta}^\mu \frac{\partial h_{n,\alpha}}{\partial t^\mu} t_X^\zeta.$$

2. The function $G(\mathbf{t})$ is known as the ‘ G -function’ of the Frobenius manifold. It was originally discovered by Getzler in his study [29] of recursion relations for genus one Gromov-Witten invariants. Indeed, for examples of Frobenius manifolds arising from quantum cohomology it is the generating function of the elliptic Gromov-Witten invariants of the symplectic manifold in question. However, its definition makes sense for an arbitrary Frobenius manifold and is non-zero for examples not arising from quantum cohomology. It has been computed explicitly for many Frobenius manifolds [62].

Example 36 (Reconstructing the KdV Hierarchy). [23] *For the Frobenius manifold corresponding to the KdV hierarchy,*

$$F = \frac{1}{6}t^3, \quad E = t \frac{\partial}{\partial t},$$

the G -function is zero, and so the perturbations $\{\cdot, \cdot\}_{[1]''}$, $\delta \mathcal{H}''$ vanish. The 4-point functions $c_{\alpha\beta\gamma\kappa}(\mathbf{t})$ also vanish because the pre-potential is cubic, meaning that the perturbation $\{\cdot, \cdot\}_{[1]'}$ also vanishes. The Hamiltonians receive the correction (5.1.8):

$$\begin{aligned} \mathcal{H}_0 &\mapsto \mathcal{H}_0 + \varepsilon^2 \delta \mathcal{H}_0' = \int_{S^1} t(X) dX, \\ \mathcal{H}_1 &\mapsto \mathcal{H}_1 + \varepsilon^2 \delta \mathcal{H}_1' = \int_{S^1} \frac{1}{6} t(X)^3 - \varepsilon^2 \frac{1}{24} t_X(X)^2 dX, \\ \mathcal{H}_2 &\mapsto \mathcal{H}_2 + \varepsilon^2 \delta \mathcal{H}_2' = \int_{S^1} \frac{1}{24} t(X)^4 - \varepsilon^2 \frac{1}{24} t(X) t_X(X)^2 dX. \end{aligned}$$

This leads to the first few flows of the hierarchy getting corrections

$$\begin{aligned} \frac{\partial t}{\partial T^0} &= t_X, \\ \frac{\partial t}{\partial T^1} &= t t_X + \frac{1}{12} \varepsilon^2 t_{XXX}, \\ \frac{\partial t}{\partial T^2} &= \frac{1}{2} t^2 t_X - \frac{1}{2} \varepsilon^2 t_X t_{XX} - \frac{1}{12} t t_{XXX}. \end{aligned}$$

5.2 Preliminaries

As pointed out above, in order to understand how the inversion symmetry acts on the first order deformation to the hierarchy, we must compute how it acts on the various objects involved in its Hamiltonian representation. The four-point functions,

$$c_{\alpha\beta\gamma\kappa}(\mathbf{t}) = \frac{\partial^4 F(\mathbf{t})}{\partial t^\alpha \partial t^\beta \partial t^\gamma \partial t^\kappa},$$

are heavily involved in the construction of these objects, and so understanding their transformation properties will be our starting point.

5.2.1 Four-point Functions in Canonical Coordinates

In canonical coordinates we replace partial derivatives with covariant ones. Therefore in canonical coordinates, the four-point functions are given by

$$c_{jkl}^i = \nabla^i \nabla_j \nabla_k \nabla_l F = \eta^{ii}(u) \nabla_i \nabla_j \nabla_k \nabla_l F.$$

Equation (1.4.1) means that locally the structure functions of our Frobenius algebras are constant:

$$c_{jk}^i = \delta_j^i \delta_k^i, \quad (5.2.1)$$

which implies that in the canonical coordinates, the four-point functions will be linear combinations of Christoffel symbols. This means that if a four-point function has more than two indices distinct, then it is zero. More concretely, by definition of the covariant derivative

$$c_{jkl}^i = 0 - \Gamma_{lj}^i \delta_k^i - \Gamma_{lk}^i \delta_j^i + \Gamma_{lj}^i \delta_k^j.$$

If $i = j = k = l$,

$$c_{iii}^i = -\Gamma_{ii}^i;$$

If $i = j = k \neq l$,

$$c_{iil}^i = -\Gamma_{il}^i;$$

If $i = j \neq k = l$,

$$c_{ill}^i = -\Gamma_{ll}^i;$$

If $i \neq j = k = l$,

$$c_{jjj}^i = \Gamma_{jj}^i.$$

It now follows from the fact that $\nabla_{wc}(X, Y, Z)$ is totally symmetric, and $\langle \partial_i, \partial_j \rangle$ diagonal that these are all non-zero 4-point functions. Now using the transformation properties of the Christoffel symbols computed (4.1.1) in Chapter 4, we can read off the transformation

properties of the four-point functions: If $i = j = k = l$, then

$$c_{iii}^i = -\Gamma_{ii}^i = -(\hat{\Gamma}_{ii}^i - \partial_i \log \hat{t}_1) = \hat{c}_{iii}^i + \partial_i \log \hat{t}_1;$$

If $i = j = k \neq l$, then

$$c_{iil}^i = -\Gamma_{il}^i = -(\hat{\Gamma}_{il}^i - \partial_l \log \hat{t}_1) = \hat{c}_{iil}^i + \partial_l \log \hat{t}_1;$$

If $i = j \neq k = l$, then

$$c_{ill}^i = -\Gamma_{ll}^i = -(\hat{\Gamma}_{ll}^i + \partial_l \log \hat{t}_1) = \hat{c}_{ill}^i - \partial_l \log \hat{t}_1;;$$

If $i \neq j = k = l$, then

$$c_{jjj}^i = \Gamma_{jj}^i = -(\hat{\Gamma}_{jj}^i + \partial_j \log \hat{t}_1) = \hat{c}_{jjj}^i + \partial_j \log \hat{t}_1.$$

5.3 Action on Deformed Hamiltonians

Together with the transformation properties of the deformed flat coordinates developed in Chapter 4, we now have essentially all the components we need to compute the action on the deformed Hamiltonians.

5.3.1 Perturbation 1: $\delta \mathcal{H}_{(n,\kappa)}'$

The first step will be to write the perturbations $\delta \mathcal{H}_{(n,\kappa)}'$ found by Dubrovin and Zhang [23] in canonical coordinates.

Proposition 19. *In canonical coordinates, the first perturbation to the Hamiltonian densities takes the form*

$$\delta \mathcal{H}_{n,\kappa} = -\frac{1}{24} \int_{S^1} \left(\sum_{r=1}^N \eta^{rr} \frac{\partial h_{(n-1,\kappa)}}{\partial u_r} \left(\frac{\partial u_r}{\partial X} \right)^2 + \sum_{r,p=1, r \neq p}^N \eta^{rr} \Gamma_{rr}^p \frac{\partial h_{(n,\kappa)}}{\partial u_p} \left(\frac{\partial u_p}{\partial X} - \frac{\partial u_r}{\partial X} \right)^2 \right) dX. \quad (5.3.1)$$

Proof. Theorem 10 states that the Hamiltonians of the principle hierarchy obtain a first correction

$$\mathcal{H}_{n,\kappa} = \int_{S^1} h_{n,\kappa}(\mathbf{t}) dX \mapsto \mathcal{H}_{n,\kappa} + \varepsilon^2 \delta \mathcal{H}_{n,\kappa}' = \int_{S^1} \left(h_{n,\kappa}(\mathbf{t}) + \varepsilon^2 \Xi_{n+1,\kappa;\mu,\nu}(\mathbf{t}) \frac{\partial t^\mu}{\partial X} \frac{\partial t^\nu}{\partial X} \right) dX,$$

where

$$\begin{aligned}\Xi_{0,\alpha;\sigma\nu}(\mathbf{t}) &= 0, \\ \Xi_{n+1,\alpha;\sigma\nu}(\mathbf{t}) &= \underbrace{\frac{1}{24} \left(c_{\mu\beta}^{\mu\sigma} c_{\nu\gamma}^{\kappa} - c_{\gamma}^{\mu\nu} c_{\mu\nu\beta}^{\kappa} \right) \frac{\partial h_{n,\alpha}}{\partial t^{\kappa}}}_{\text{term A}} - \underbrace{\frac{1}{24} c_{\xi\xi}^{\gamma} c_{\nu}^{\xi\sigma} c_{\sigma\mu}^{\xi} \frac{\partial h_{n-1,\alpha}}{\partial t^{\gamma}}}_{\text{term B}}.\end{aligned}$$

Consider term A. In canonical coordinates, this becomes

$$\frac{1}{24} \sum_{s,l,m,r,n=1}^N (c_{sm}^{sl} c_{ln}^r \frac{\partial h_{n,\kappa}}{\partial u_r}) \frac{\partial u_m}{\partial X} \frac{\partial u_n}{\partial X} - \frac{1}{24} \sum_{a,b,r,n,m=1}^N (c_n^{ab} c_{abm}^r \frac{\partial h_{n,\kappa}}{\partial u_r}) \frac{\partial u_m}{\partial X} \frac{\partial u_n}{\partial X}.$$

Using (5.2.1) this simplifies to

$$\frac{1}{24} \sum_{m,r,s=1}^N c_{sm}^{sr} \frac{\partial h_{n,\kappa}}{\partial u_r} \frac{\partial u_m}{\partial X} \frac{\partial u_r}{\partial X} - \sum_{r,m,b=1}^N c_{bm}^{br} \frac{\partial h_{n,\kappa}}{\partial u_r} \frac{\partial u_m}{\partial X} \frac{\partial u_b}{\partial X} =: \text{term A}|_u.$$

As pointed out above, the four-point functions in canonical coordinates are linear combinations of Christoffel symbols, and so if they have more than three distinct indices they vanish. Therefore we decompose the above sums as follows:

$$\begin{aligned}\text{term A}|_u &= \frac{1}{24} \left(\sum_{s,m=1,s \neq m}^N c_{sm}^{ss} \frac{\partial h_{n,\kappa}}{\partial u_s} \frac{\partial u_m}{\partial X} \frac{\partial u_s}{\partial X} + \sum_{s,n=1,s \neq n}^N c_{ss}^{sn} \frac{\partial h_{n,\kappa}}{\partial u_n} \frac{\partial u_n}{\partial X} \frac{\partial u_s}{\partial X} \right. \\ &\quad + \sum_{s,m=1,s \neq n}^N c_{sm}^{sm} \frac{\partial h_{n,\kappa}}{\partial u_m} \left(\frac{\partial u_m}{\partial X} \right)^2 + \sum_{s=1}^N c_{ss}^{ss} \frac{\partial h_{n,\kappa}}{\partial u_s} \left(\frac{\partial u_s}{\partial X} \right)^2 \Big) \\ &\quad - \frac{1}{24} \left(\sum_{s,m=1,s \neq m}^N c_{sm}^{ss} \frac{\partial h_{n,\kappa}}{\partial u_s} \frac{\partial u_m}{\partial X} \frac{\partial u_s}{\partial X} + \sum_{s,n=1,s \neq n}^N c_{ss}^{sn} \frac{\partial h_{n,\kappa}}{\partial u_n} \left(\frac{\partial u_s}{\partial X} \right)^2 \right. \\ &\quad + \sum_{s,m=1,s \neq n}^N c_{sm}^{sm} \frac{\partial h_{n,\kappa}}{\partial u_m} \frac{\partial u_m}{\partial X} \frac{\partial u_s}{\partial X} + \sum_{s=1}^N c_{ss}^{ss} \frac{\partial h_{n,\kappa}}{\partial u_s} \left(\frac{\partial u_s}{\partial X} \right)^2 \Big) \\ &= \frac{1}{24} \left(\sum_{s,m=1,s \neq m}^N c_{sm}^{sm} \frac{\partial h_{n,\kappa}}{\partial u_m} \frac{\partial u_s}{\partial X} \left(\frac{\partial u_m}{\partial X} - \frac{\partial u_s}{\partial X} \right) + \sum_{s,m=1,s \neq m}^N c_{sm}^{sm} \frac{\partial h_{n,\kappa}}{\partial u_m} \frac{\partial u_m}{\partial X} \left(\frac{\partial u_m}{\partial X} - \frac{\partial u_s}{\partial X} \right) \right) \\ &= \frac{1}{24} \left(\sum_{s,m=1,s \neq m}^N \eta^{ss} c_{ssm}^m \frac{\partial h_{n,\kappa}}{\partial u_m} \frac{\partial u_s}{\partial X} \left(\frac{\partial u_m}{\partial X} - \frac{\partial u_s}{\partial X} \right) \right. \\ &\quad + \sum_{s,m=1,s \neq m}^N \eta^{ss} c_{ssm}^m \frac{\partial h_{n,\kappa}}{\partial u_m} \frac{\partial u_m}{\partial X} \left(\frac{\partial u_m}{\partial X} - \frac{\partial u_s}{\partial X} \right) \Big) \\ &= \sum_{s,m=1,s \neq m}^N \eta^{ss} \Gamma_{ss}^m \frac{\partial h_{n,\kappa}}{\partial u_m} \frac{\partial u_s}{\partial X} \left(\frac{\partial u_m}{\partial X} - \frac{\partial u_s}{\partial X} \right) - \sum_{s,m=1,s \neq m}^N \eta^{ss} \Gamma_{ss}^m \frac{\partial h_{n,\kappa}}{\partial u_m} \frac{\partial u_m}{\partial X} \left(\frac{\partial u_m}{\partial X} - \frac{\partial u_s}{\partial X} \right) \\ &= -\frac{1}{24} \sum_{s,m=1,s \neq m}^N \eta^{ss} \Gamma_{ss}^m \frac{\partial h_{(n,\kappa)}}{\partial u_m} \left(\frac{\partial u_m}{\partial X} - \frac{\partial u_s}{\partial X} \right)^2. \tag{5.3.2}\end{aligned}$$

Now consider term B. This term is a lot simpler since it contains no 4–point functions. In flat coordinates we have

$$\text{term B}|_t = -\frac{1}{24} c_{\lambda\xi}^\gamma \eta^{\lambda\varepsilon} c_{\varepsilon\nu}^\sigma c_{\sigma\mu}^\xi \frac{\partial h_{n-1,\kappa}}{\partial t^\gamma} \frac{dt^\mu}{dX} \frac{dt^\nu}{dX}.$$

Again canonical coordinates simplify this expression,

$$\begin{aligned} \text{term B}|_u &= -\frac{1}{24} \sum_{r,l,i,s,n,m=1}^N (\delta_l^r \delta_i^r) \eta^{ll} (\delta_l^s \delta_n^s) (\delta_s^i \delta_m^i) \frac{\partial h_{n-1,\kappa}}{\partial u_r} \frac{du_m}{dX} \frac{du_n}{dX} \\ &= -\frac{1}{24} \sum_{i=1}^N \eta^{ii} \frac{\partial h_{n-1,\kappa}}{\partial u_i} \left(\frac{du_i}{dX} \right)^2. \end{aligned}$$

Hence

$$\text{term A}|_u + \text{term B}|_u = -\frac{1}{24} \sum_{s,m=1, s \neq m}^N \eta^{ss} \Gamma_{ss}^m \frac{\partial h_{(n,\kappa)}}{\partial u_m} \left(\frac{\partial u_m}{\partial X} - \frac{\partial u_s}{\partial X} \right)^2 - \frac{1}{24} \sum_{s=1}^N \eta^{ss} \frac{\partial h_{n-1,\kappa}}{\partial u_s} \left(\frac{du_s}{dX} \right)^2,$$

which is equal to the (integrand appearing on the) right hand side. \square

Now we have an expression for the first perturbation in canonical coordinates, we can apply the inversion symmetry.

Corollary 4. *Under the inversion symmetry, the first perturbation to the Hamiltonians transforms as*

$$\delta \mathcal{H}'_{n,\kappa} = \pm \hat{t}_1 \delta \hat{\mathcal{H}}'_{\tilde{n},\tilde{\kappa}} + \hat{\#}'_{\tilde{n},\tilde{\alpha}} \quad (5.3.3)$$

where

$$\hat{\#}'_{\tilde{n},\tilde{\alpha}} = \pm \frac{1}{24} \int_{S^1} \left(\sum_{s,m=1, s \neq m}^N \left(-\hat{h}_{\tilde{n},\tilde{\kappa}} (\hat{\Gamma}_{sm}^s + \partial_m \log \hat{t}_1) + \frac{\partial \hat{h}_{\tilde{n},\tilde{\kappa}}}{\partial u_m} \right) \left(\frac{\partial u_m}{\partial X} - \frac{\partial u_s}{\partial X} \right)^2 \pm \hat{h}_{\tilde{n}-1,\tilde{\kappa}} \sum_{s=1}^N \left(\frac{\partial u_s}{\partial X} \right)^2 \right) dX,$$

unless $(n, \kappa) = (0, 1)$, in which case

$$\delta \mathcal{H}'_{0,1} = \delta \hat{\mathcal{H}}'_{0,1} - \frac{1}{24} \int_{S^1} \sum_{r,i=1, r \neq i}^N \partial_r \log \hat{t}_1 \left(\frac{\partial u_r}{\partial X} - \frac{\partial u_i}{\partial X} \right)^2 dX.$$

Proof. Recall the transformation properties for the unperturbed Hamiltonians:

$$h_{n,\kappa}(\mathbf{t}(\hat{\mathbf{t}})) = \pm \frac{1}{\hat{t}_1} \hat{h}_{\tilde{n},\tilde{\kappa}}(\hat{\mathbf{t}}); \quad \tilde{n} = \begin{cases} n+1, & \text{if } \alpha = N, \\ n, & \text{if } \alpha \neq 1, N, \\ n-1, & \text{if } \alpha = 1, \end{cases} \quad \tilde{\alpha} = \begin{cases} 1, & \text{if } \alpha = N, \\ \alpha, & \text{if } \alpha \neq 1, N, \\ N, & \text{if } \alpha = 1. \end{cases}$$

We use these, combined with the relationship between Christoffel symbols corresponding

to a pair of metrics related by a conformal transformation. We have

$$\begin{aligned}\delta\mathcal{H}'_{n,\kappa} &= -\frac{1}{24} \int_{S^1} \left(\sum_{r=1}^N \eta^{rr} \frac{\partial h_{(n-1,\kappa)}}{\partial u_r} \left(\frac{\partial u_r}{\partial X} \right)^2 + \sum_{r,p=1, r \neq p}^N \Gamma_{rr}^p \frac{\partial h_{(n,\kappa)}}{\partial u_p} \left(\frac{\partial u_p}{\partial X} - \frac{\partial u_r}{\partial X} \right)^2 \right) dX \\ &= -\frac{1}{24} \int_{S^1} \sum_{s \neq m} \hat{t}_1^2 \hat{\eta}^{ss} (\hat{\Gamma}_{ss}^m + \partial_s \log \hat{t}_1) \left(\frac{\pm 1}{\hat{t}_1} \frac{\partial \hat{h}_{\bar{n}, \bar{\kappa}}}{\partial u_m} \mp \frac{\hat{h}_{\bar{n}, \bar{\kappa}}}{\hat{t}_1^2} \partial_m \hat{t}_1 \right) \left(\frac{\partial u_p}{\partial X} - \frac{\partial u_r}{\partial X} \right)^2 dX \\ &\quad - \frac{1}{24} \int_{S^1} \hat{t}_1^2 \sum_{s=1}^N \hat{\eta}^{ss} \left(\pm \frac{1}{\hat{t}_1} \frac{\partial \hat{h}_{\bar{n}-1, \bar{\kappa}}}{\partial u_s} \mp \frac{\hat{h}_{\bar{n}-1, \bar{\kappa}}}{\hat{t}_1^2} \partial_s \hat{t}_1 \right) \left(\frac{\partial u_s}{\partial X} \right)^2 dX. \quad (5.3.4)\end{aligned}$$

Now using the fact that \hat{t}_1 is the Egoroff potential for the metric $\hat{\eta}$, $\partial_s \hat{t}_1 = \hat{\eta}_{ss}$, the above formula (5.3.3) follows immediately. For the exceptional case $(n, \kappa) = (0, 1)$, note that the perturbation takes the form

$$\delta\mathcal{H}'_{0,1} = \frac{1}{24} \int_{S^1} c^{\sigma\mu\nu} \frac{\partial t^\mu}{\partial X} \frac{\partial t^\nu}{\partial X} dX = -\frac{1}{24} \int_{S^1} \sum_{r,i=1, r \neq i}^N c_{iri}^i \left(\frac{\partial u_r}{\partial X} - \frac{\partial u_i}{\partial X} \right)^2 dX.$$

From the transformation properties of the 4-point functions the result follows immediately. \square

5.3.2 Perturbation 2: $\delta\mathcal{H}''_{(n,\kappa)}$

The inclusion of a G -function also gives a correction to the Hamiltonians, whose transformation properties will again be investigated in the canonical coordinate system. A key ingredient in understanding the behaviour of this perturbation is the transformation properties of the G -function under the inversion symmetry. This was computed for an arbitrary semi-simple Frobenius manifold by Strachan [63]:

$$G = \hat{G} + \left(\frac{N}{24} - \frac{1}{2} \right) \log \hat{t}_1.$$

Thus we have all the information we need to compute the action of the inversion symmetry on this second perturbation.

Proposition 20. *In canonical coordinates, the second perturbation to the Hamiltonians takes the form*

$$\delta\mathcal{H}''_{n,\kappa} = \int_{S^1} \sum_{s=1}^N \eta^{-1} \left(\frac{\partial h_{n,\kappa}}{\partial u_s}, \frac{\partial G}{\partial u_s} \right) \left(\frac{\partial u_s}{\partial X} \right)^2 dX, \quad (5.3.5)$$

where η^{-1} is the metric on $T^*\mathcal{M}$ induced by η .

Proof. This follows from the fact that in canonical coordinates the multiplication diagonalises. \square

Corollary 5. *Under the inversion symmetry, the perturbation $\delta \mathcal{H}_{n,\kappa}''$ transforms as*

$$\delta \mathcal{H}_{n,\kappa}'' = \pm \hat{t}_1 \delta \hat{\mathcal{H}}_{n,\kappa}'' + \hat{\#}_{\tilde{n},\tilde{\alpha}}'' \quad (5.3.6)$$

where

$$\hat{\#}_{\tilde{n},\tilde{\alpha}}'' = \mp \int_{S^1} \sum_{s=1}^N \left(\hat{h}_{\tilde{n},\tilde{\kappa}} \frac{\partial \hat{G}}{\partial u_s} - \left(\frac{\partial \hat{h}_{\tilde{n},\tilde{\kappa}}}{\partial u_s} - \hat{h}_{\tilde{n},\tilde{\kappa}} \partial_s \log \hat{t}_1 \right) \left(\frac{N}{24} - \frac{1}{2} \right) \right) \left(\frac{\partial u_s}{\partial X} \right)^2 dX,$$

unless $(n, \kappa) = (0, 1)$, in which case

$$\begin{aligned} \delta \mathcal{H}_{0,1}'' &= \delta \hat{\mathcal{H}}_{0,1}'' + \left(\frac{N}{24} - \frac{1}{2} \right) \int_{S^1} \sum_{i=1}^N \partial_i \log \hat{t}_1 \left(\frac{\partial u_i}{\partial X} \right)^2 dX \\ &= \delta \hat{\mathcal{H}}_{0,1}'' + \left(\frac{N}{24} - \frac{1}{2} \right) \int_{S^1} \frac{1}{\hat{t}_1} \frac{\partial \hat{t}_\sigma}{\partial X} \frac{\partial \hat{t}^\sigma}{\partial X} dX. \end{aligned} \quad (5.3.7)$$

Proof. Combining the above proposition with the transformation properties of the G -function gives

$$\begin{aligned} & \sum_{s=1}^N \eta^{-1} \left(\frac{\partial h_{n,\kappa}}{\partial u_s}, \frac{\partial G}{\partial u_s} \right) \left(\frac{\partial u_s}{\partial X} \right)^2 \\ &= \sum_{s=1}^N \hat{t}_1^2 \hat{\eta}^{-1} \left(\pm \frac{1}{\hat{t}_1} \frac{\partial h_{n,\kappa}}{\partial u_s} \mp \frac{\hat{h}_{\tilde{n},\tilde{\kappa}}}{\hat{t}_1^2} \partial_s \hat{t}_1, \frac{\partial \hat{G}}{\partial u_s} + \left(\frac{N}{24} - \frac{1}{2} \right) \partial_s \log \hat{t}_1 \right) \left(\frac{\partial u_s}{\partial X} \right)^2 \\ &= \pm \hat{t}_1 \sum_{i=1}^N \hat{\eta}^{ii} \frac{\partial \hat{h}_{\tilde{n},\tilde{\kappa}}}{\partial u_i} \frac{\partial \hat{G}}{\partial u_i} \left(\frac{du_i}{dX} \right)^2 \mp \hat{h}_{\tilde{n},\tilde{\kappa}} \sum_{i=1}^N \frac{\partial \hat{G}}{\partial u_i} \left(\frac{du_i}{dX} \right)^2 \\ &\quad \pm \left(\frac{N}{24} - \frac{1}{2} \right) \sum_{i=1}^N \hat{\eta}^{ii} \frac{\partial \hat{h}_{\tilde{n},\tilde{\kappa}}}{\partial u_i} \left(\frac{du_i}{dX} \right)^2 \mp \left(\frac{N}{24} - \frac{1}{2} \right) \frac{1}{\hat{t}_1} \hat{h}_{\tilde{n},\tilde{\kappa}} \sum_{i=1}^N \hat{\eta}^{ii} \left(\frac{du_i}{dX} \right)^2. \end{aligned}$$

For the exceptional case $(n, \kappa) = (0, 1)$, note that the perturbation takes the form

$$\delta \mathcal{H}_{0,1}'' = \int_{S^1} \eta_{\xi\nu} c_\mu^{\sigma\xi} \frac{\partial G}{\partial t^\sigma} \frac{\partial t^\mu}{\partial X} \frac{\partial t^\nu}{\partial X} dX = \int_{S^1} \sum_{i=1}^N \frac{\partial G}{\partial u_i} \left(\frac{\partial u_i}{\partial X} \right)^2 dX.$$

Using the transformation properties of the G -function, we have

$$\begin{aligned} \delta \mathcal{H}_{0,1}'' &= \int_{S^1} \sum_{i=1}^N \left(\frac{\partial \hat{G}}{\partial u_i} + \left(\frac{N}{24} - \frac{1}{2} \right) \partial_i \log \hat{t}_1 \right) \left(\frac{\partial u_i}{\partial X} \right)^2 dX \\ &= \delta \hat{\mathcal{H}}_{0,1}'' + \left(\frac{N}{24} - \frac{1}{2} \right) \int_{S^1} \sum_{i=1}^N \partial_i \log \hat{t}_1 \left(\frac{\partial u_i}{\partial X} \right)^2 dX. \end{aligned}$$

□

5.4 Action on Deformed Poisson Prackets

5.4.1 Deformations of the Poisson Brackets in Canonical Coordinates

We follow the ideas of Dubrovin & Zhang, repeating their arguments for the construction of the perturbations but this time in canonical coordinates. They construct the perturbations in flat coordinates, homogeneous of degree three. Thus, in general we may assume that the general form of the perturbation in canonical coordinates is of the form

$$P^{ij}(u) = P_{[0]}^{ij}(\mathbf{u}) + \varepsilon^2 \left(P_{[1]'}^{ij}(\mathbf{u}) + P_{[1]''}^{ij}(\mathbf{u}) \right) + \mathcal{O}(\varepsilon^4).$$

Proposition 21. *In canonical coordinates, the first perturbations to the first Poisson bracket takes the form*

$$P_{[1]'}^{ij}(\mathbf{u}) = f^{ij}(\mathbf{u}) \nabla_X^3 + \frac{3}{2} \nabla_X(f^{ij}(\mathbf{u})) \nabla_X^2 + \frac{1}{2} \nabla_X^2(f^{ij}(\mathbf{u})) \nabla_X, \quad (5.4.1)$$

where

$$\sum_{i,j} f^{ij}(\mathbf{u}) \frac{\partial}{\partial u_i} \otimes \frac{\partial}{\partial u_j} = \sum_{i,j,r,s} \eta^{rs} c_{rs}^{ij} \frac{\partial}{\partial u_i} \otimes \frac{\partial}{\partial u_j} \in \Gamma(T\mathcal{M} \otimes T\mathcal{M}, \mathcal{M}).$$

Proof. Recall first that in the flat coordinates this perturbation takes the form

$$P_{[1]'}^{\alpha\beta}(\mathbf{t}) = \frac{\varepsilon^2}{24} \left(f^{\alpha\beta}(\mathbf{t}) \partial_X^3 + \frac{3}{2} \partial_X(f^{\alpha\beta}(\mathbf{t})) \partial_X^2 + \frac{1}{2} \partial_X^2(f^{\alpha\beta}(\mathbf{t})) \partial_X \right).$$

This follows from antisymmetry of the Poisson bracket

$$\{\mathcal{F}, \mathcal{G}\} = \int_{S^1} \frac{\delta \mathcal{F}}{\delta t^\alpha} P_{[1]'}^{\alpha\beta} \frac{\delta \mathcal{G}}{\delta t^\beta} dX \quad (5.4.2)$$

and that the functionals

$$\int_{S^1} t^\gamma(X) dX + \mathcal{O}(\varepsilon^4)$$

span its centre. Secondly, in order for (5.4.2) to define a Poisson bracket, we must have that $f^{\alpha\beta}(\mathbf{t})$ are the components of a tensor of rank $(2,0)$. Finally, we view the operator ∂_X as the covariant derivative along the vector field tangent to the image of a loop in our manifold, written in flat coordinates. Thus in another coordinate system where the Christoffel symbols are non-zero, we must replace this operator with the appropriate covariant derivative:

$$\partial_X \longrightarrow \nabla_X. \quad (5.4.3)$$

For example,

$$\begin{aligned}\partial_X f^{\alpha\beta}(\mathbf{t}) &= \partial_X \left(\sum_{i,j} \frac{\partial t^\alpha}{\partial u_i} \frac{\partial t^\beta}{\partial u_j} f^{ij}(\mathbf{u}(\mathbf{t})) \right) \\ &= \sum_{i,j} \left(\frac{\partial t^\alpha}{\partial u_i} \frac{\partial t^\beta}{\partial u_j} \partial_X (f^{ij}(\mathbf{u}(\mathbf{t}))) + \partial_X \left(\frac{\partial t^\alpha}{\partial u_i} \right) \frac{\partial t^\beta}{\partial u_j} f^{ij}(\mathbf{u}(\mathbf{t})) + \frac{\partial t^\alpha}{\partial u_i} \partial_X \left(\frac{\partial t^\beta}{\partial u_j} \right) f^{ij}(\mathbf{u}(\mathbf{t})) \right).\end{aligned}\quad (5.4.4)$$

Now

$$\partial_X \left(\frac{\partial t^\alpha}{\partial u_i} \right) = \sum_k \frac{\partial^2 t^\alpha}{\partial u_i \partial u_k} \frac{\partial u_k}{\partial X} = \sum_{j,k} \frac{\partial t^\alpha}{\partial u_j} \Gamma_{ik}^j \frac{\partial u_k}{\partial X},$$

so we can rewrite the above expression (5.4.4) as

$$\begin{aligned}\sum_{i,j} \left(\frac{\partial t^\alpha}{\partial u_i} \frac{\partial t^\beta}{\partial u_j} \partial_X (f^{ij}(\mathbf{u}(\mathbf{t}))) + \frac{\partial t^\beta}{\partial u_j} \sum_{r,s} \frac{\partial t^\alpha}{\partial u_r} \Gamma_{is}^r \frac{\partial u_s}{\partial X} + \frac{\partial t^\alpha}{\partial u_i} \sum_{r,s} \frac{\partial t^\beta}{\partial u_r} \Gamma_{js}^r \frac{\partial u_s}{\partial X} \right) \\ = \sum_{i,j} \frac{\partial t^\alpha}{\partial u_i} \frac{\partial t^\beta}{\partial u_j} \nabla_X (f^{ij}(\mathbf{u}(\mathbf{t}))).\end{aligned}\quad (5.4.5)$$

The covariant derivative along a fixed vector field is a derivation

$$\nabla_X : \Gamma(T\mathcal{M}^{\otimes r} \otimes T^*\mathcal{M}^{\otimes s}) \rightarrow \Gamma(T\mathcal{M}^{\otimes r} \otimes T^*\mathcal{M}^{\otimes s})$$

that preserves the rank of tensors. So the array $\nabla_X(f^{ij})$ still defines a tensor of rank two, although its components may now depends on the jets:

$$\sum_{i,j} \nabla_X(f^{ij}(\mathbf{u})) \frac{\partial}{\partial u_i} \otimes \frac{\partial}{\partial u_j} = \sum_{i,j} (\tilde{f}^{ij}(\mathbf{u}, \partial_X \mathbf{u})) \frac{\partial}{\partial u_i} \otimes \frac{\partial}{\partial u_j}$$

Therefore we may also replace the higher order derivatives as well:

$$\begin{aligned}\partial_X^2 &\longrightarrow \nabla_X^2, \\ \partial_X^3 &\longrightarrow \nabla_X^3.\end{aligned}$$

□

Corollary 6. *In the canonical coordinates, the flows of the principal hierarchy do not diagonalize.*

Proof. This follows from the fact that the tensor $f^{ij}(\mathbf{u})$ is not diagonal in the canonical coordinates. □

Remark. Dubrovin's approach to the theory of Poisson brackets of this form is different in the following way: He considers Poisson brackets of the form

$$\{\mathcal{H}, \mathcal{F}\} = \int_{S^1} \sum_{\alpha, \beta=1}^N \left(\frac{\delta \mathcal{H}}{\delta v^\alpha} \right) \sum_{s=1}^N A_s^{\alpha\beta} \partial_X^s \left(\frac{\delta \mathcal{F}}{\delta v^\beta} \right) dX$$

where the coefficients $A_s^{\alpha\beta}$ are *not* tensors for $s \neq N$. As a consequence, under a change of coordinates their transformations properties are extremely complicated. In my formalism, we replace the operators ∂_X with covariant ones ∇_X , where ∂_X is understood as a vector field on the manifold, that is

$$\{\mathcal{H}, \mathcal{F}\} = \int_{S^1} \sum_{\alpha, \beta=1}^N \left(\frac{\delta \mathcal{H}}{\delta v^\alpha} \right) \sum_{s=1}^N A_s^{\alpha\beta} \nabla_X^s \left(\frac{\delta \mathcal{F}}{\delta v^\beta} \right) dX.$$

Here all the coefficients $A_s^{\alpha\beta}$ are tensors of rank $(2,0)$. The reason for doing this is that when we come to look at the action of the inversion symmetry, this will make it easier to compute.

We repeat the idea for the second deformation arising from the inclusion of a G -function. In the case just considered all the coefficients in the Hamiltonian operator $P_{[1]'}^{ij}(\mathbf{u})$ are functions of $f^{ij}(\mathbf{u})$. For the second perturbation this is not the case.

Proposition 22. *In canonical coordinates, the second perturbations to the first Poisson bracket takes the form*

$$P_{[1]''}^{ij}(\mathbf{u}) = a^{ij}(\mathbf{u}) \nabla_X^3 + \left(\frac{3}{2} \nabla_X(a^{ij}(\mathbf{u})) + b^{ij}(\mathbf{u}, \partial_X \mathbf{u}) \right) \nabla_X^2 + (\nabla_X b^{ij}(\mathbf{u}, \partial_X \mathbf{u}) + \frac{1}{2} \nabla_X^2(a^{ij}(\mathbf{u}))) \nabla_X, \quad (5.4.6)$$

where

$$a^{ij}(\mathbf{u}) = 2\delta^{ij}(\eta^{ii})^2 \frac{\partial G}{\partial \mathbf{u}_j}, \quad b^{ij}(\mathbf{u}, \partial_X \mathbf{u}) = \nabla^i \nabla^j G \left(\frac{\partial u_i}{\partial X} - \frac{\partial u_j}{\partial X} \right), \quad (5.4.7)$$

and we use the abbreviation $\nabla^i = \eta^{ii} \nabla_i$.

Proof. In the flat coordinate system, Dubrovin and Zhang showed that this second perturbation takes the form

$$P_{[1]''}^{\alpha\beta}(\mathbf{t}) = a^{\alpha\beta}(\mathbf{t}) \partial_X^3 + \left(\frac{3}{2} \partial_X(a^{\alpha\beta}(\mathbf{t})) + b^{\alpha\beta}(\mathbf{t}, \partial_X \mathbf{t}) \right) \partial_X + (\partial_X b^{\alpha\beta}(\mathbf{t}, \partial_X \mathbf{t}) + \partial_X^2(a^{\alpha\beta}(\mathbf{t}))) \partial_X$$

where

$$a^{\alpha\beta}(\mathbf{t}) = c^{\alpha\beta\gamma} \frac{\partial G}{\partial t^\gamma}, \quad b^{\alpha\beta}(\mathbf{t}, \partial_X \mathbf{t}) = \frac{\partial^2 G}{\partial t^\sigma \partial t^\rho} (c_\mu^{\alpha\sigma} \eta^{\beta\rho} - c_\mu^{\beta\sigma} \eta^{\alpha\rho}) \frac{\partial t^\mu}{\partial X}.$$

The expression (5.4.6) then follows in an analogous manner to the calculations carried out in the proof of Proposition 21. \square

5.4.2 The Inversion Symmetry and the Deformed Poisson Brackets

Now that we have expressions for the Poisson brackets in the canonical coordinate system we can start to compute how the inversion symmetry acts on them. Just as in the proof of Proposition 15 presented in Chapter 4, firstly, we must compute how the objects $f^{ij}(\mathbf{u})$, $a^{ij}(\mathbf{u})$, and $b^{ij}(\mathbf{u})$ behave under the inversion symmetry. The second step will be to consider the behaviour of their covariant derivatives.

Recall that

$$f = \sum_{i,j} f^{ij}(\mathbf{u}) \frac{\partial}{\partial u_i} \otimes \frac{\partial}{\partial u_j} = \frac{1}{12} \sum_{i,j,r,s} \eta^{rs} c_{rs}^{ij} \frac{\partial}{\partial u_i} \otimes \frac{\partial}{\partial u_j} \in \Gamma(T\mathcal{M} \otimes T\mathcal{M}, \mathcal{M}).$$

We need consider the transformation properties of $f^{ij}(\mathbf{u})$ in two stages, according to whether we are considering diagonal components of $f^{ij}(\mathbf{u})$ or not. Firstly, if $i \neq j$ we have

$$\begin{aligned} \frac{1}{12} \sum_r c_r^{ijr} &= \frac{1}{12} (c_i^{iji} + c_j^{ijj}) = \frac{1}{12} \eta^{ii} \eta^{jj} (c_{iji}^i + c_{ijj}^j) \\ &= \frac{1}{12} \eta^{ii} \eta^{jj} (-\Gamma_{ij}^i - \Gamma_{ji}^j) \\ &= \frac{\hat{t}_1^4}{12} \hat{\eta}^{ii} \hat{\eta}^{jj} (-\hat{\Gamma}_{ij}^i - \hat{\Gamma}_{ji}^j) + \frac{\hat{t}_1^4}{12} (\partial_i \log \hat{t}_1 + \partial_j \log \hat{t}_1) \\ &= \hat{t}_1^4 \hat{f}^{ij} + \frac{\hat{t}_1^3}{12} (\hat{\eta}^{ii} + \hat{\eta}^{jj}) \end{aligned} \tag{5.4.8}$$

In the case $i = j$, we have

$$\begin{aligned} f^{ii} = \frac{1}{12} \sum_s c_s^{iis} &= \frac{1}{12} (\eta^{ii})^2 \sum_s c_{is}^s = \frac{1}{12} (\eta^{ii})^2 \left(\sum_{s \neq i} (c_{iis}^s) + c_{iii}^i \right) \\ &= \frac{1}{12} (\eta^{ii})^2 \left(\sum_{r \neq i} (-\Gamma_{ii}^r) - \Gamma_{ii}^i \right) = \frac{1}{12} (\eta^{ii})^2 \left(\sum_{r \neq i} (\Gamma_{ri}^r) - \Gamma_{ii}^i \right) \\ &= \frac{\hat{t}_1^4}{12} (\hat{\eta}^{ii})^2 \left(\sum_{r \neq i} (\hat{\Gamma}_{ri}^r - \partial_i \log \hat{t}_1) - \hat{\Gamma}_{ii}^i + \partial_i \log \hat{t}_1 \right) \\ &= \hat{t}_1^4 \hat{f}^{ii} - \frac{\hat{t}_1^4}{12} (N-2) (\hat{\eta}^{ii})^2 \partial_i \log \hat{t}_1. \end{aligned}$$

Therefore

$$f^{ij}(\mathbf{u}) = \hat{t}_1^4 \hat{f}^{ij}(\mathbf{u}) + \hat{t}_1^3 \hat{\phi}^{ij}(\mathbf{u}), \tag{5.4.9}$$

where

$$\hat{\phi}^{ij}(\mathbf{u}) = \begin{cases} \frac{1}{12} (\hat{\eta}^{ii} + \hat{\eta}^{jj}), & \text{for } i \neq j, \\ \frac{1}{12} (2-N) \hat{\eta}^{ii}, & \text{for } i = j. \end{cases}$$

It will be useful to also have an expression for the transformation properties in the flat

coordinate system. Let us denote

$$\hat{\phi}^{\alpha\beta} = \sum_{i,j=1}^N \frac{\partial \hat{t}^\alpha}{\partial u_i} \frac{\partial \hat{t}^\beta}{\partial u_j} \hat{\phi}^{ij}.$$

Un-packing this we have

$$\begin{aligned} \sum_{i,j=1}^N \frac{\partial \hat{t}^\alpha}{\partial u_i} \frac{\partial \hat{t}^\beta}{\partial u_j} \hat{\phi}^{ij} &= \sum_{i=1}^N \frac{\partial \hat{t}^\alpha}{\partial u_i} \frac{\partial \hat{t}^\beta}{\partial u_i} \hat{\phi}^{ii} + \sum_{i \neq j} \frac{\partial \hat{t}^\alpha}{\partial u_i} \frac{\partial \hat{t}^\beta}{\partial u_j} \hat{\phi}^{ij} \\ &= \frac{1}{12} \left((2-N) \hat{\eta}^{\alpha\beta} + \sum_{i=1}^N \frac{\partial \hat{t}^\alpha}{\partial u_i} \left(\hat{e}(\hat{t}^\beta) - \frac{\partial \hat{t}^\beta}{\partial u_i} \right) \hat{\eta}^{ii} + \sum_{j=1}^N \left(\hat{e}(\hat{t}^\alpha) - \frac{\partial \hat{t}^\alpha}{\partial u_j} \right) \frac{\partial \hat{t}^\beta}{\partial u_j} \hat{\eta}^{jj} \right) \\ &= -\frac{N}{12} \hat{\eta}^{\alpha\beta} + \frac{1}{12} \sum_{j=1}^N \hat{\eta}^{jj} \frac{\partial}{\partial u_j} (\delta^{\alpha 1} \hat{t}^\beta + \delta^{\beta 1} \hat{t}^\alpha) \\ &= -\frac{N}{12} \hat{\eta}^{\alpha\beta} + \frac{1}{12} \left(\delta^{\alpha 1} \frac{\partial}{\partial \hat{t}_\beta} + \delta^{\beta 1} \frac{\partial}{\partial \hat{t}_\alpha} \right) \sum_{i=1}^N u_i. \end{aligned} \quad (5.4.10)$$

Now commutativity of the diagram

$$\begin{array}{ccc} f^{ij}(u) & \xrightarrow{I} & \hat{t}_1^4 \hat{f}^{ij}(u) + \hat{t}_1^3 \hat{\phi}^{ij}(u) \\ \downarrow u \mapsto t & & \downarrow u \mapsto \hat{t} \\ \frac{\partial u_i}{\partial \hat{t}^\alpha} \frac{\partial u_j}{\partial \hat{t}^\beta} f^{\alpha\beta}(t) & \xrightarrow[\hat{I}]{} & \frac{\partial u_i}{\partial \hat{t}^\alpha} \frac{\partial u_j}{\partial \hat{t}^\beta} (\hat{t}_1^4 \hat{f}^{\alpha\beta}(\hat{t}) + \hat{t}_1^3 \hat{\phi}^{\alpha\beta}(\hat{t})) \end{array} \quad (5.4.11)$$

means that in the flat coordinate system we have

$$f^{\alpha\beta} = \frac{\partial t^\alpha}{\partial \hat{t}^\epsilon} \frac{\partial t^\beta}{\partial \hat{t}^\kappa} (\hat{t}_1^4 \hat{f}^{\epsilon\kappa} + \hat{t}_1^3 \hat{\phi}^{\epsilon\kappa}), \quad (5.4.12)$$

where $\hat{\phi}^{\epsilon\kappa}$ are defined by (5.4.10). This follows from the fact that the canonical coordinate system is fixed by the inversion symmetry (recall that the action in canonical coordinates is by a conformal transformation of the metric η). There is a check we can perform to make sure the calculation is correct. In the flat coordinates the tensor $f^{\alpha\beta} = 0$ if $\alpha = N$, or $\beta = N$, which follows from the fact that $c_{1\alpha\beta}$ are constants. Now by definition of the inversion symmetry, t_1 is a function of \hat{t}_1 alone, and so

$$\frac{\partial t^N}{\partial \hat{t}^\epsilon} = \frac{\delta_\epsilon^N}{(t^N)^2}.$$

Therefore

$$f^{N\alpha} = 0 \Rightarrow \hat{\phi}^{N\alpha} = 0. \quad (5.4.13)$$

We need only check the case $(\alpha, \beta) = (N, 1)$ since the others will be satisfied trivially by the definition (5.4.10). We have

$$\hat{\phi}^{N1} = -\frac{N}{12} + \frac{1}{12} \sum_{i=1}^N \frac{\partial \hat{t}^N}{\partial u_i} \hat{\eta}^{ii} = -\frac{N}{12} + \frac{1}{12} \sum_{i=1}^N \hat{\eta}_{ii} \hat{\eta}^{ii} = 0,$$

since \hat{t}^N is the Egoroff potential for the metric $\hat{\eta}$. So the identity (5.4.13) is satisfied.

We also take note of the analogous statement in canonical coordinates. Namely the endomorphism $f_j^i(u)$ induced by lowering one index using the metric is trace free:

$$\sum_{s=1}^N \hat{\phi}^{rs} \partial_s \hat{t}_1 = 0, \quad \sum_{s=1}^N \hat{f}^{rs} \partial_s \hat{t}_1 = 0. \quad (5.4.14)$$

To see prove the first assertion, we simply un-pack the definition of the tail term $\hat{\phi}^{ij}$. We have

$$\begin{aligned} \sum_{s=1}^N \hat{\phi}^{rs} \partial_s \hat{t}_1 &= \sum_{s \neq r} \frac{1}{12} \left(\frac{1}{\partial_r \hat{t}_1} + \frac{1}{\partial_s \hat{t}_1} \right) \partial_s \hat{t}_1 + \frac{2-N}{12} \frac{\partial_r \hat{t}_1}{\partial_r \hat{t}_1} \\ &= \sum_{s \neq r} \frac{1}{12} \left(\frac{\partial_s \hat{t}_1}{\partial_r \hat{t}_1} + 1 \right) + \frac{2-N}{12} \\ &= (N-1) \frac{1}{12} - \frac{1}{12} + \frac{2-N}{12} = 0, \end{aligned}$$

since $\hat{e}(\hat{t}_1) = 0$. To prove the second assertion we write out the expression for the tensor f^{ij} in canonical coordinates. We have

$$\begin{aligned} \sum_{s=1}^N \hat{f}^{rs} \partial_s \hat{t}_1 &= \sum_{s,i=1}^N \hat{c}_i^{irs} \partial_s \hat{t}_1 = \sum_{i,s} \hat{c}_{is}^{ir} = \sum_i \hat{\eta}^{ii} \left(\sum_s \hat{c}_{is}^r \right) \\ &= \sum_i \hat{\eta}^{ii} \left(\sum_s \hat{\Gamma}_{is}^r \right) = 0, \end{aligned}$$

as $\hat{\nabla}(\hat{e}) = 0 \Rightarrow \sum_s \hat{\Gamma}_{is}^r = 0$. The identities (5.4.14) will be useful when we come to let the operator act on the Hamiltonians, since the components of their exterior derivatives transform according to

$$\frac{\partial h_{\alpha,p}(\mathbf{u})}{\partial u_s} \mapsto \pm \frac{1}{\hat{t}_1} \frac{\partial \hat{h}_{\tilde{\alpha},\tilde{p}}(\mathbf{u})}{\partial u_s} \mp \frac{\partial_s \hat{t}_1}{\hat{t}_1^2} \hat{h}_{\tilde{\alpha},\tilde{p}}(\mathbf{u})$$

under the inversion symmetry. Using the identities (5.4.14) together with $\hat{\nabla}_X \partial_s \hat{t}_1 = 0$ it also follows that

$$\sum_{s=1}^N \hat{\nabla}_X^n (\hat{\phi}^{rs}) \partial_s \hat{t}_1 = 0, \quad \sum_{s=1}^N \hat{\nabla}_X^n (\hat{h}^{rs}) \partial_s \hat{t}_1 = 0, \quad \text{for } n \geq 0: \quad (5.4.15)$$

By (5.4.14), we have, for example:

$$0 = \hat{\nabla}_X \left(\sum_{s=1}^N \hat{\phi}^{rs} \partial_s \hat{t}_1 \right) = \hat{\nabla}_X \left(\sum_{s=1}^N \hat{\phi}^{rs} \right) \partial_s \hat{t}_1 + \sum_{s=1}^N \hat{\phi}^{rs} \underbrace{\hat{\nabla}_X (\partial_s \hat{t}_1)}_{=0}.$$

The other identities follow analogously.

For the second perturbation, $P_{[1]''}^{ij}(\mathbf{u})$, we must understand the action on the objects

$$a^{ij}(\mathbf{u}) = 2\delta^{ij}(\eta^{ii})^2 \frac{\partial G}{\partial u_j}, \quad b^{ij}(\mathbf{u}, \partial_X \mathbf{u}) = \nabla^i \nabla^j G \frac{\partial}{\partial X} (u_i - u_j).$$

A quick calculation gives

$$a^{ij}(\mathbf{u}) = \hat{t}_1^4 \hat{a}^{ij}(\mathbf{u}) + \hat{t}_1^3 \hat{\eta}^{ij} \left(\frac{N}{12} - 1 \right) =: \hat{t}_1^4 \hat{a}^{ij}(\mathbf{u}) + \hat{t}_1^3 \hat{\psi}^{ij}(\mathbf{u}). \quad (5.4.16)$$

Therefore in the flat coordinate system we have

$$a^{\alpha\beta} = \frac{\partial t^\alpha}{\partial \hat{t}^\sigma} \frac{\partial t^\beta}{\partial \hat{t}^\kappa} (\hat{t}_1^4 \hat{a}^{\sigma\kappa} + \hat{t}_1^3 \hat{\psi}^{\sigma\kappa}) \quad \text{for } \hat{\psi}^{\alpha\beta}(\hat{t}) := \sum_{i,j=1}^N \frac{\partial \hat{t}^\alpha}{\partial u_i} \frac{\partial \hat{t}^\beta}{\partial u_j} \hat{\psi}^{ij}(u). \quad (5.4.17)$$

Explicitly,

$$\hat{\psi}^{\alpha\beta} = \left(\frac{N}{12} - 1 \right) \hat{\eta}^{\alpha\beta}$$

The calculation of the transformation properties of the $b^{ij}(\mathbf{u})$ is slightly more involved.

First off,

$$G = \hat{G} + \left(\frac{N}{24} - \frac{1}{2} \right) \log \hat{t}_1 \Rightarrow dG = d\hat{G} + \left(\frac{N}{24} - \frac{1}{2} \right) d \log \hat{t}_1,$$

and so

$$\nabla_j \left(\frac{\partial G}{\partial u_i} \right) = \frac{\partial^2 \hat{G}}{\partial u_i \partial u_j} - \sum_s \Gamma_{ij}^s \frac{\partial \hat{G}}{\partial u_s} + \left(\frac{N}{24} - \frac{1}{2} \right) \nabla_j \left(\frac{1}{\hat{t}_1} \partial_i \hat{t}_1 \right).$$

Now use the transformation properties for the Christoffel symbols, and the fact that b^{ij} is anti-symmetric (this means we only need the transformation properties of Γ_{ij}^i and Γ_{ij}^j)

in the canonical coordinates to get

$$\begin{aligned}
\nabla_i \nabla_j G &= \hat{\nabla}_i \hat{\nabla}_j \hat{G} + \partial_i \log \hat{t}_1 \partial_j \hat{G} + \partial_j \log \hat{t}_1 \partial_i \hat{G} \\
&\quad + \left(\frac{N}{24} - \frac{1}{2} \right) \left\{ \partial_j \left(\frac{\hat{\eta}_{ii}}{\hat{t}_1} \right) - \Gamma_{ij}^s \left(\frac{\hat{\eta}_{ii}}{\hat{t}_1} \right) \right\} \\
&= \hat{\nabla}_i \hat{\nabla}_j \hat{G} + \partial_i \log \hat{t}_1 \partial_j \hat{G} + \partial_j \log \hat{t}_1 \partial_i \hat{G} + \left(\frac{N}{24} - \frac{1}{2} \right) \\
&\quad \times \left\{ -\frac{1}{\hat{t}_1^2} \hat{\eta}_{jj} \hat{\eta}_{ii} - \underbrace{\hat{\Gamma}_{ij}^s \left(\frac{\hat{\eta}_{ss}}{\hat{t}_1} \right) + \frac{1}{\hat{t}_1} \partial_j \hat{\eta}_{ii}}_{=0 \text{ as } \hat{\nabla} \hat{\eta} = 0} + \partial_i \log \hat{t}_1 \frac{\hat{\eta}_{jj}}{\hat{t}_1} + \partial_j \log \hat{t}_1 \frac{\hat{\eta}_{ii}}{\hat{t}_1} \right\} \quad (5.4.18)
\end{aligned}$$

And so, using the fact that covariant differentiation commutes with the raising and lowering of indices, we arrive at

$$\begin{aligned}
b^{ij}(\mathbf{u}, \partial_X \mathbf{u}) &= \hat{t}_1^4 \hat{b}^{ij}(\mathbf{u}, \partial_X \mathbf{u}) + \left\{ \hat{t}_1^3 (\hat{\eta}^{jj} \partial_j \hat{G} + \hat{\eta}^{ii} \partial_i \hat{G}) \right. \\
&\quad + \left. \hat{t}_1^4 \hat{\eta}^{ii} \hat{\eta}^{jj} \left(\frac{N}{24} - \frac{1}{2} \right) \left(-\frac{1}{\hat{t}_1^2} \hat{\eta}_{jj} \hat{\eta}_{ii} + \partial_i \log \hat{t}_1 \frac{\hat{\eta}_{jj}}{\hat{t}_1} + \partial_j \log \hat{t}_1 \frac{\hat{\eta}_{ii}}{\hat{t}_1} \right) \right\} \frac{\partial}{\partial X} (u_i - u_j) \\
&= \hat{t}_1^4 \hat{b}^{ij}(\mathbf{u}, \partial_X \mathbf{u}) + \left\{ \hat{t}_1^3 (\hat{\eta}^{jj} \partial_j \hat{G} + \hat{\eta}^{ii} \partial_i \hat{G}) + \hat{t}_1^2 \left(\frac{N}{24} - \frac{1}{2} \right) \right\} \frac{\partial}{\partial X} (u_i - u_j) \\
&=: \hat{t}_1^4 \hat{b}^{ij}(\mathbf{u}) + \hat{\omega}^{ij}(\mathbf{u}). \quad (5.4.19)
\end{aligned}$$

To find the rule for the transformation properties of $b^{\alpha\beta}(\mathbf{t})$, i.e. an explicit formula for

$$\hat{\omega}^{\alpha\beta}(\hat{\mathbf{t}}) := \sum_{i,j=1}^N \frac{\partial \hat{t}^\alpha}{\partial u_i} \frac{\partial \hat{t}^\beta}{\partial u_j} \hat{\omega}^{ij}(\mathbf{u}),$$

consider the equation (5.4.18) in flat coordinates:

$$\frac{\partial^2 G}{\partial t^\sigma \partial t^\kappa} = \frac{\partial^2 \hat{G}}{\partial \hat{t}^\sigma \partial \hat{t}^\kappa} + \frac{\partial \log \hat{t}_1}{\partial \hat{t}^\sigma} \frac{\partial \hat{G}}{\partial \hat{t}^\kappa} + \frac{\partial \log \hat{t}_1}{\partial \hat{t}^\kappa} \frac{\partial \hat{G}}{\partial \hat{t}^\sigma} + \left(\frac{N}{24} - \frac{1}{2} \right) \frac{\partial \log \hat{t}_1}{\partial \hat{t}^\sigma} \frac{\partial \log \hat{t}_1}{\partial \hat{t}^\kappa}.$$

Using this we can compute directly the transformation properties of the $b^{\alpha\beta}(\mathbf{t})$:

$$\begin{aligned}
b^{\alpha\beta}(\mathbf{t}) &= \frac{\partial^2 G}{\partial t^\sigma \partial t^\rho} (c_\mu^{\alpha\sigma} \eta^{\beta\rho} - c_\mu^{\beta\sigma} \eta^{\alpha\rho}) \frac{\partial t^\mu}{\partial X} \\
&= \frac{\partial t^\alpha}{\partial \hat{t}^\varepsilon} \frac{\partial t^\beta}{\partial \hat{t}^\nu} \left\{ \left(\frac{\partial^2 \hat{G}}{\partial \hat{t}^\sigma \partial \hat{t}^\kappa} + \frac{\partial \log \hat{t}_1}{\partial \hat{t}^\sigma} \frac{\partial \hat{G}}{\partial \hat{t}^\kappa} + \frac{\partial \log \hat{t}_1}{\partial \hat{t}^\kappa} \frac{\partial \hat{G}}{\partial \hat{t}^\sigma} + \left(\frac{N}{24} - \frac{1}{2} \right) \frac{\partial \log \hat{t}_1}{\partial \hat{t}^\sigma} \frac{\partial \log \hat{t}_1}{\partial \hat{t}^\kappa} \right) \right. \\
&\quad \times \left. \hat{t}_1^4 (\hat{c}_\mu^{\varepsilon\sigma} \hat{\eta}^{\nu\rho} - \hat{c}_\mu^{\nu\sigma} \hat{\eta}^{\varepsilon\rho}) \frac{\partial \hat{t}^\mu}{\partial X} \right\} \\
&= \frac{\partial t^\alpha}{\partial \hat{t}^\varepsilon} \frac{\partial t^\beta}{\partial \hat{t}^\nu} (\hat{t}_1^4 \hat{b}^{\varepsilon\nu} + \hat{\omega}^{\varepsilon\nu}),
\end{aligned}$$

for

$$\hat{\omega}^{\varepsilon\nu} = \hat{t}_1^3 \left((\delta_\mu^\varepsilon \hat{\eta}^{\nu\rho} - \delta_\mu^\nu \hat{\eta}^{\varepsilon\rho}) \frac{\partial \hat{G}}{\partial \hat{t}^\rho} + (\delta^{\nu 1} \hat{c}_\mu^{\varepsilon\sigma} - \delta^{\varepsilon 1} \hat{c}_\mu^{\nu\sigma}) \frac{\partial \hat{G}}{\partial \hat{t}^\sigma} \right) \frac{\partial \hat{t}^\mu}{\partial X} + \hat{t}_1^2 \left(\frac{N}{24} - \frac{1}{2} \right) \left(\delta^{\nu 1} \frac{\partial \hat{t}^\varepsilon}{\partial X} - \delta^{\varepsilon 1} \frac{\partial \hat{t}^\nu}{\partial X} \right).$$

Let us now turn our attention to the behaviour of the covariant derivatives.

$\nabla_X f^{ij}(\mathbf{u})$. Since this is the first computation of this nature we'll go through it step by step. Firstly, we compute the covariant derivatives of h along the basic vector fields $\partial/\partial u_i$. Because of the index-dependence of the transformation properties of the Christoffel symbols, we must decompose the calculations accordingly. Consider first the off-diagonal components of h in the canonical coordinate system. Suppose s, i and j are all distinct.

$$\begin{aligned} \nabla_s f^{ij} &= \partial_s f^{ij} + \sum_{k=1}^N (\Gamma_{sk}^i h^{kj} + \Gamma_{sk}^j h^{ik}) \\ &= \partial_s f^{ij} + \Gamma_{ss}^i f^{sj} + \Gamma_{si}^i f^{ij} + \Gamma_{sj}^j f^{ij} + \Gamma_{ss}^j f^{is} \\ &= \partial_s f^{ij} + (\hat{\Gamma}_{ss}^i + \partial_s \log \hat{t}_1) f^{sj} + (\hat{\Gamma}_{si}^i - \partial_s \log \hat{t}_1) f^{ij} \\ &\quad + (\hat{\Gamma}_{sj}^j - \partial_s \log \hat{t}_1) f^{ij} + (\hat{\Gamma}_{ss}^j + \partial_s \log \hat{t}_1) f^{is} \\ &= \hat{\nabla}_s f^{ij} + \partial_s \log \hat{t}_1 (f^{sj} + f^{is} - 2f^{ij}). \end{aligned}$$

We also have

$$\begin{aligned} \nabla_i f^{ij} &= \partial_i f^{ij} + \sum_{k=1}^N (\Gamma_{ik}^i h^{kj} + \Gamma_{ik}^j h^{ik}) \\ &= \partial_i f^{ij} + \Gamma_{ii}^i f^{ij} + \sum_{k \neq i} \Gamma_{ik}^i h^{kj} + \Gamma_{ij}^j f^{ij} + \Gamma_{ii}^j f^{ii} \\ &= \partial_i f^{ij} + (\hat{\Gamma}_{ii}^i - \partial_i \log \hat{t}_1) f^{ij} + \sum_{k \neq i} (\hat{\Gamma}_{ik}^i - \partial_k \log \hat{t}_1) f^{kj} \\ &\quad + (\hat{\Gamma}_{ij}^j - \partial_i \log \hat{t}_1) f^{ij} + (\hat{\Gamma}_{ii}^j + \partial_i \log \hat{t}_1) f^{ii} \\ &= \hat{\nabla}_i f^{ij} - \sum_{k=1}^N \partial_k \log \hat{t}_1 h^{kj} + \partial_i \log \hat{t}_1 (f^{ii} - f^{ij}) \\ &= \hat{\nabla}_i f^{ij} + \partial_i \log \hat{t}_1 (f^{ii} - f^{ij}) \quad \text{by identity (5.4.14)} \end{aligned} \tag{5.4.20}$$

Therefore, for $i \neq j$

$$\begin{aligned}
\nabla_X f^{ij} &= \sum_{s=1}^N \frac{\partial u_s}{\partial X} \nabla_s f^{ij} \\
&= \hat{\nabla}_X f^{ij} + \sum_{s \neq i,j} \frac{\partial u_s}{\partial X} \partial_s \log \hat{t}_1 (f^{sj} + f^{is} - 2f^{ij}) \\
&\quad + \frac{\partial u_i}{\partial X} \partial_i \log \hat{t}_1 (f^{ii} - f^{ij}) + \frac{\partial u_j}{\partial X} \partial_j \log \hat{t}_1 (f^{jj} - f^{ij}) \\
&= \hat{\nabla}_X f^{ij} - 2\partial_X \log \hat{t}_1 f^{ij} + \frac{1}{\hat{t}_1} \frac{\partial u_i}{\partial X} \partial_i \hat{t}_1 (f^{ij} + f^{ii}) + \frac{1}{\hat{t}_1} \frac{\partial u_j}{\partial X} \partial_j \hat{t}_1 (f^{ij} + f^{jj}) \\
&\quad + \frac{1}{\hat{t}_1} \sum_{s \neq i,j} \frac{\partial u_s}{\partial X} \partial_s \hat{t}_1 (f^{sj} + f^{is}) \\
&= \hat{\nabla}_X f^{ij} - 2\partial_X \log \hat{t}_1 f^{ij} + \frac{1}{\hat{t}_1} \sum_{s=1}^N \frac{\partial u_s}{\partial X} \partial_s \hat{t}_1 (f^{sj} + f^{is}). \tag{5.4.21}
\end{aligned}$$

The case $i = j$ is easier since there are less sub-cases to consider. Quick calculations give

$$\begin{aligned}
\nabla_s f^{ii} &= \hat{\nabla}_s f^{ii} + 2\partial_s \log \hat{t}_1 f^{si} - 2\partial_s \log \hat{t}_1 f^{ii}, \\
\nabla_i f^{ii} &= \hat{\nabla}_i f^{ii} - 2\partial_i \log \hat{t}_1 f^{ii} - 2 \sum_{s \neq i} \partial_s \log \hat{t}_1 f^{si}
\end{aligned}$$

which imply

$$\begin{aligned}
\nabla_X f^{ii} &= \hat{\nabla}_X f^{ii} - 2\partial_X \log \hat{t}_1 f^{ii} + 2 \sum_{s \neq i} \frac{\partial u_s}{\partial X} \partial_s \log \hat{t}_1 f^{si} - 2 \frac{\partial u_i}{\partial X} \sum_{s \neq i} \partial_s \log \hat{t}_1 f^{si} \\
&= \hat{\nabla}_X f^{ii} - 2\partial_X \log \hat{t}_1 f^{ii} + 2 \sum_{s=1}^N \frac{\partial u_s}{\partial X} \partial_s \log \hat{t}_1 f^{si} \quad \text{by identity (5.4.14)} \tag{5.4.22}
\end{aligned}$$

In conclusion

$$\nabla_X f^{ij} = \hat{\nabla}_X f^{ij} - 2\partial_X \log \hat{t}_1 f^{ij} + \frac{1}{\hat{t}_1} \left(\sum_{s=1}^N \frac{\partial u_s}{\partial X} \partial_s \hat{t}_1 (f^{is} + f^{sj}) \right). \tag{5.4.23}$$

Now compose this with the transformation law (5.4.9) to get

$$\nabla_X f^{ij} = \hat{t}_1^4 \hat{\nabla}_X \hat{f}^{ij} + \hat{t}_1^3 \hat{\nabla}_X \hat{\phi}^{ij} + 2\hat{t}_1^3 \partial_X \hat{t}_1 \hat{f}^{ij} + \hat{t}_1^2 \partial_X \hat{t}_1 \hat{\phi}^{ij} + \frac{1}{\hat{t}_1} \sum_{s=1}^N \frac{\partial u_s}{\partial X} \partial_s \hat{t}_1 (\hat{t}_1^4 (\hat{f}^{sj} + \hat{f}^{is}) + \hat{t}_1^3 (\hat{\phi}^{sj} + \hat{\phi}^{is})).$$

Expanding,

$$\hat{t}_1^3 \sum_{s=1}^N \frac{\partial u_s}{\partial X} (\hat{f}_s^j + \hat{f}_s^i) = \hat{t}_1^3 \left(\hat{\nabla}_X \hat{\phi}^{ij} - \frac{N}{12} \delta^{ij} \underbrace{\hat{\nabla}_X \hat{\eta}^{ii}}_{=0} \right),$$

$$\begin{aligned}
\hat{t}_1^2 \sum_{s=1}^N \frac{\partial u_s}{\partial X} \partial_s \hat{t}_1 (\hat{\phi}^{is} + \hat{\phi}^{sj}) &= \frac{\hat{t}_1^2}{12} (\hat{\eta}^{ii} + \hat{\eta}^{jj}) \partial_X \hat{t}_1 + \frac{\hat{t}_1^2}{6} \partial_X \left(\sum_{s=1}^N u_s \right) - \frac{N \hat{t}_1^2}{12} \partial_X (u_i + u_j) \\
&= \hat{t}_1^2 \partial_X \hat{t}_1 \hat{\phi}^{ij} + \frac{\hat{t}_1^2}{6} \partial_X \left(\sum_{s=1}^N u_s \right) - \frac{N \hat{t}_1^2}{12} (\partial_X (u_i + u_j) - \delta^{ij} \hat{\eta}^{ii} \partial_X \hat{t}_1).
\end{aligned}$$

Hence

$$\begin{aligned}
\nabla_X f^{ij} &= \hat{t}_1^4 \left(\hat{\nabla}_X + 2 \partial_X \log \hat{t}_1 \right) \hat{f}^{ij} + 2 \hat{t}_1^3 \left(\hat{\nabla}_X + \partial_X \log \hat{t}_1 \right) \hat{\phi}^{ij} \\
&\quad + \frac{\hat{t}_1^2}{6} \partial_X \left(\sum_{s=1}^N u_s \right) - \frac{N \hat{t}_1^2}{12} (\partial_X (u_i + u_j) - \delta^{ij} \hat{\eta}^{ii} \partial_X \hat{t}_1). \tag{5.4.24}
\end{aligned}$$

In the flat coordinate system, this reads

$$\begin{aligned}
\partial_X f^{\alpha\beta} &= \frac{\partial t^\alpha}{\partial \hat{t}^\sigma} \frac{\partial t^\beta}{\partial \hat{t}^\varepsilon} \left(\hat{t}_1^4 (\partial_X + 2 \partial_X \log \hat{t}_1) \hat{h}^{\sigma\varepsilon} + \frac{\hat{t}_1^3}{6} (\partial_X + \partial_X \log \hat{t}_1) \left\{ \delta^{\sigma 1} \frac{\partial}{\partial \hat{t}_\varepsilon} + \delta^{\varepsilon 1} \frac{\partial}{\partial \hat{t}_\sigma} \right\} \sum_{i=1}^N u_i \right. \\
&\quad \left. + \frac{N \hat{t}_1^2}{12} \left\{ \delta^{\varepsilon 1} \frac{\partial \hat{t}^\sigma}{\partial X} + \delta^{\sigma 1} \frac{\partial \hat{t}^\varepsilon}{\partial X} - \partial_X \hat{t}_1 \hat{\eta}^{\varepsilon\sigma} \right\} \right) + \frac{\hat{t}_1^2}{6} \partial_X \sum_{i=1}^N u_i. \tag{5.4.25}
\end{aligned}$$

The appearance of terms of the form $\sum_i u_i$ is intriguing. In the flat coordinate system it has the following interpretation. Let $\mathcal{U} : T\mathcal{M} \rightarrow T\mathcal{M}$, $X \mapsto E \circ X$. Then the canonical coordinates are the eigenvalues of this operator, and the trace is their sum:

$$\sum_{i=1}^N u_i = \text{tr}(\mathcal{U}_v^\varepsilon); \quad \mathcal{U}_v^\varepsilon := E^\sigma c_{\sigma v}^\varepsilon.$$

$\nabla_X a^{ij}(\mathbf{u})$. Somewhat simpler calculations (recall that a diagonalizes in the canonical coordinates) give

$$\begin{aligned}
\nabla_s a^{ii} &= \hat{\nabla}_s a^{ii} - 2 \partial_s \log \hat{t}_1 a^{ii}, \quad \text{for } s \neq i, \\
\nabla_i a^{ii} &= \hat{\nabla}_i a^{ii} - 2 \partial_i \log \hat{t}_1 a^{ii}.
\end{aligned}$$

Hence

$$\nabla_X a^{ij} = \hat{t}_1^4 \hat{\nabla}_X \hat{a}^{ij} + 2 \hat{t}_1^3 \partial_X \hat{t}_1 \hat{a}^{ij} + \hat{t}_1^2 \partial_X \hat{t}_1 \hat{\psi}^{ij}. \tag{5.4.26}$$

5.4.3 The Action on the Sections $\nabla_X dh_{\alpha,p}$ and $\nabla_X^2 dh_{\alpha,p}$.

The final ingredient we need is the action on the 1-forms $\nabla_X dh_{\alpha,p}$ and $\nabla_X^2 dh_{\alpha,p}$. Again, we will use canonical coordinates to do the calculations.

$\nabla_X dh_{\alpha,p}$. Recall that the Hamiltonian densities satisfy the recursion relations

$$\nabla_i \nabla_j h_{\alpha,p} = \delta_i^k \delta_j^k \nabla_k h_{\alpha,p-1}. \quad (5.4.27)$$

We use this to obtain in canonical coordinates

$$\begin{aligned} \nabla_X \left(\frac{\partial h_{\alpha,p}}{\partial u_i} \right) &= \sum_{s=1}^N \frac{\partial u_s}{\partial X} \nabla_s \nabla_i h_{\alpha,p} \\ &= \frac{\partial u_i}{\partial X} \nabla_i h_{\alpha,p-1} \quad \text{using (5.4.27)}. \end{aligned} \quad (5.4.28)$$

Note the particular case

$$\frac{\partial u_i}{\partial X} \partial_{i1} = \nabla_X \left(\frac{\partial h_{1,1}}{\partial u_i} \right). \quad (5.4.29)$$

Using the transformation properties of the exterior derivatives $dh_{\alpha,p}$ together with (5.4.28) it is straightforward to compute the action on $\nabla_X dh_{\alpha,p}$:

$$\begin{aligned} \frac{\partial u_i}{\partial X} \nabla_i h_{\alpha,p-1} &= \frac{\partial u_i}{\partial X} \left(\pm \frac{1}{\hat{t}_1} \hat{\nabla}_i \hat{h}_{\tilde{\alpha},\tilde{p}-1} \mp \frac{1}{\hat{t}_1^2} \partial_{i1} \hat{h}_{\tilde{\alpha},\tilde{p}-1} \right) \\ &= \pm \frac{1}{\hat{t}_1} \hat{\nabla}_X \left(\frac{\partial h_{\tilde{\alpha},\tilde{p}}}{\partial u_i} \right) \mp \frac{1}{\hat{t}_1^2} \hat{\nabla}_X \left(\frac{\partial h_{1,1}}{\partial u_i} \right) h_{\tilde{\alpha},\tilde{p}-1}, \end{aligned} \quad (5.4.30)$$

where in the last step we have used the identity (5.4.29). The coordinate free expression is:

$$\nabla_X (dh_{\alpha,p}) = \pm \frac{1}{\hat{t}_1} \hat{\nabla}_X (d\hat{h}_{\tilde{\alpha},\tilde{p}}) \mp \frac{1}{\hat{t}_1^2} \hat{\nabla}_X (d\hat{h}_{1,1}) \hat{h}_{\tilde{\alpha},\tilde{p}-1} \quad (5.4.31)$$

This agrees with the earlier results when we investigated the action of the inversion symmetry on the dispersionless principal hierarchy. The above expression is essentially those transformation properties.

$\nabla_X^2 dh_{\alpha,p}$. We first obtain an expression for the i^{th} component of the section $\nabla_X^2 dh_{\alpha,p}$ in canonical coordinates and then apply the inversion symmetry. Using the recursion relation (5.4.27) we have

$$\begin{aligned} \nabla_X^2 (\partial_i h_{\alpha,p}) &= \nabla_X \left(\frac{\partial u^i}{\partial X} \nabla_i h_{\alpha,p-1} \right) \\ &= \nabla_X^2 u_i \nabla_i h_{\alpha,p-1} + \partial_X u_i \nabla_X \nabla_i h_{\alpha,p-1} \\ &= \nabla_X^2 u_i \nabla_i h_{\alpha,p-1} + (\partial_X u_i)^2 \nabla_i h_{\alpha,p-2}. \end{aligned} \quad (5.4.32)$$

Hence we need to calculate the transformation properties of $\nabla_X^2 u_i$. We treat $\partial_X u_i$ as a vector field and compute its covariant derivative accordingly, and so as above the transformation properties of the Christoffel symbols will then dictate the transformation properties of $\nabla_X^2 u_i$. Due to the index dependence of the transformation properties

of the Christoffel symbols we will decompose the calculation as follows. Let $v = \sum_i v^i \partial_i \in \Gamma(T\mathcal{M}, \mathcal{M})$. Suppose $s \neq i$. Then

$$\begin{aligned} \nabla_s v^i &= \partial_s v^i + \sum_{p=1}^N \Gamma_{ps}^i v^p \\ &= \partial_s v^i + \Gamma_{is}^i v^i + \Gamma_{ss}^i v^s \\ &= \partial_s v^i + (\hat{\Gamma}_{is}^i - \partial_s \log \hat{t}_1) v^i + (\Gamma_{ss}^i + \partial_s \log \hat{t}_1) v^s \\ &= \hat{\nabla}_s v^i + \partial_s \log \hat{t}_1 (v^s - v^i). \end{aligned}$$

If $s = i$ then

$$\begin{aligned} \nabla_i v^i &= \partial_i v^i + \sum_{p=1}^N \Gamma_{pi}^i v^p \\ &= \partial_i v^i + \Gamma_{ii}^i v^i + \sum_{p \neq i} \Gamma_{pi}^i v^p \\ &= \partial_i v^i + (\hat{\Gamma}_{ii}^i - \partial_i \log \hat{t}_1) v^i + \sum_{p \neq i} (\hat{\Gamma}_{pi}^i - \partial_p \log \hat{t}_1) v^p \\ &= \hat{\nabla}_i v^i - \partial_i \log \hat{t}_1 v^i - \sum_{p \neq i} \partial_p \log \hat{t}_1 v^p. \end{aligned}$$

Hence

$$\begin{aligned} \nabla_X v^i &= \hat{\nabla}_X v^i + \sum_{s \neq i} \partial_X u_s (\partial_s \log \hat{t}_1 (v^s - v^i)) + \partial_X u_i \left(-\partial_i \log \hat{t}_1 v^i - \sum_{s \neq i} \partial_s \log \hat{t}_1 v^s \right) \\ &= \hat{\nabla}_X v^i - \partial_X \log \hat{t}_1 v^i + \sum_{s \neq i} \partial_s \log \hat{t}_1 v^s (\partial_X (u_s - u_i)). \end{aligned} \quad (5.4.33)$$

Now, for the particular case $v^i = \partial_X u_i$, we get

$$\nabla_X^2 u_i = \hat{\nabla}_X^2 u_i - 2 \partial_X \log \hat{t}_1 \partial_X u_i + \sum_{s=1}^n \partial_s \log \hat{t}_1 (\partial_X u_s)^2. \quad (5.4.34)$$

We can re-write this as

$$\nabla_X^2 u_i = \hat{\nabla}_X^2 u_i - 2 \partial_X \log \hat{t}_1 \partial_X u_i + \frac{1}{\hat{t}_1} \hat{\eta}(\partial_X \mathbf{u}, \partial_X \mathbf{u}).$$

Now we have all the ingredients to invert (5.4.32). We get

$$\begin{aligned}
\nabla_X^2(\partial_i h_{\alpha,p}) &= \left(\hat{\nabla}_X^2 u_i - 2\partial_X \log \hat{t}_1 \partial_X u_i + \frac{1}{\hat{t}_1} \hat{\eta}(\partial_X u, \partial_X u) \right) \left(\pm \frac{1}{\hat{t}_1} \hat{\nabla}_i \hat{h}_{\bar{\alpha}, \bar{p}-1} \mp \frac{1}{\hat{t}_1^2} \partial_i \hat{t}_1 \hat{h}_{\bar{\alpha}, \bar{p}-1} \right) \\
&\quad + (\partial_X u_i)^2 \left(\pm \frac{1}{\hat{t}_1} \hat{\nabla}_i \hat{h}_{\bar{\alpha}, \bar{p}-2} \mp \frac{1}{\hat{t}_1^2} \partial_i \hat{t}_1 \hat{h}_{\bar{\alpha}, \bar{p}-2} \right) \\
&= \pm \frac{1}{\hat{t}_1} \left(\hat{\nabla}_X - 2\partial_X \log \hat{t}_1 \right) \hat{\nabla}_X (\partial_i \hat{h}_{\bar{\alpha}, \bar{p}}) \mp \frac{\hat{h}_{\bar{\alpha}, \bar{p}-1}}{\hat{t}_1^2} \left(\hat{\nabla}_X - 2\partial_X \log \hat{t}_1 \right) \hat{\nabla}_X (\partial_i \hat{h}_{1,1}) \\
&\quad + \hat{\eta}(\partial_X u, \partial_X u) \left(\pm \frac{1}{\hat{t}_1} \partial_i \hat{h}_{\bar{\alpha}, \bar{p}-1} \mp \frac{1}{\hat{t}_1^2} \partial_i \hat{t}_1 \hat{h}_{\bar{\alpha}, \bar{p}-1} \right) \mp \frac{1}{\hat{t}_1^2} \partial_X u_i \hat{\nabla}_X (\partial_i \hat{h}_{1,1}) \hat{h}_{\bar{\alpha}, \bar{p}} \quad (5.4.35)
\end{aligned}$$

The first two terms appear to mimic those of (5.4.31), while the meaning of the latter two is more mysterious.

Confirmation of the action on ∇_X . We will show how to confirm (5.4.31) using the conformal geometry approach. Let us recall briefly how the covariant derivatives of two one-forms are related via the inversion symmetry. Let $\phi = \sum_i \phi_i du_i \in \Gamma(T^* \mathcal{M}, \mathcal{M})$. Suppose $i \neq j$.

$$\begin{aligned}
\nabla_j \phi_i &= \partial_j \phi_i - \sum_{s=1}^N \Gamma_{ji}^s \phi_s = \partial_j \phi_i - \Gamma_{ji}^j \phi_j - \Gamma_{ji}^i \phi_i \\
&= \partial_j \phi_i = \partial_j \phi_i - \left(\hat{\Gamma}_{ji}^j - \partial_j \log \hat{t}_1 \right) \phi_j - \left(\hat{\Gamma}_{ji}^i - \partial_j \log \hat{t}_1 \right) \phi_i \\
&= \hat{\nabla}_j \phi_i + \partial_i \log \hat{t}_1 \phi_j + \partial_j \log \hat{t}_1 \phi_i. \quad (5.4.36)
\end{aligned}$$

If $i = j$, we have

$$\begin{aligned}
\nabla_i \phi_i &= \partial_i \phi_i - \sum_{s=1}^N \Gamma_{ii}^s \phi_s = \partial_i \phi_i - \Gamma_{ii}^i \phi_i = \sum_{s \neq i} \Gamma_{ii}^s \phi_s \\
&= \partial_i \phi_i - \left(\hat{\Gamma}_{ii}^i - \partial_i \log \hat{t}_1 \right) \phi_i - \sum_{s \neq i} \left(\hat{\Gamma}_{ii}^s + \partial_i \log \hat{t}_1 \right) \phi_s \\
&= \hat{\nabla}_i \phi_i + \partial_i \log \hat{t}_1 \phi_i - \partial_i \log \hat{t}_1 \sum_{s \neq i} \phi_s. \quad (5.4.37)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla_X \phi_i &= \sum_{s=1}^N \frac{\partial u_s}{\partial X} \nabla_s \phi_i = \frac{\partial u_i}{\partial X} \nabla_i \phi_i + \sum_{s \neq i} \frac{\partial u_s}{\partial X} \nabla_s \phi_i \\
&= \hat{\nabla}_X \phi_i + \frac{\partial u_i}{\partial X} \left(\partial_i \log \hat{t}_1 \phi_i - \partial_i \log \hat{t}_1 \sum_{s \neq i} \phi_s \right) + \sum_{s \neq i} (\partial_i \log \hat{t}_1 \phi_s + \partial_s \log \hat{t}_1 \phi_i) \frac{\partial u_s}{\partial X} \\
&= \hat{\nabla}_X \phi_i + \partial_X \log \hat{t}_1 \phi_i + \partial_i \log \hat{t}_1 \sum_{s \neq i} \phi_s \frac{\partial}{\partial X} (u_s - u_i). \quad (5.4.38)
\end{aligned}$$

From this calculation we just obtain the transformation properties of the operator $\hat{\nabla}_X$ acting on 1-forms.

Consider the case $\phi_i = \partial_i h_{\alpha,p}$. Using the transformation properties of the functions $\{h_{\alpha,p}\}$, and the fact that $e(h_{\alpha,p}) = h_{\alpha,p-1}$, we have

$$\sum_{s \neq i} \partial_s h_{\alpha,p} = h_{\alpha,p-1} - \partial_i h_{\alpha,p},$$

and the expression (5.4.38) becomes:

$$\nabla_X (\partial_i h_{\alpha,p}) = \hat{\nabla}_X (\partial_i h_{\alpha,p}) + \partial_X \log \hat{t}_1 \partial_i h_{\alpha,p} + \partial_i \log \hat{t}_1 \partial_X h_{\alpha,p} - \frac{\partial u_i}{\partial X} \partial_i \log \hat{t}_1 h_{\alpha,p-1}. \quad (5.4.39)$$

Composition with the transformation properties of the $\{h_{\alpha,p}\}$ gives:

$$\begin{aligned} \hat{\nabla}_X (\partial_i h_{\alpha,p}) &= \hat{\nabla}_X \left(\pm \frac{1}{\hat{t}_1} \partial_i \hat{h}_{\tilde{n},\tilde{\alpha}} \mp \frac{1}{\hat{t}_1^2} \partial_i \hat{t}_1 \hat{h}_{\tilde{n},\tilde{\alpha}} \right) \\ &= \pm \frac{1}{\hat{t}_1} \hat{\nabla}_X \hat{h}_{\tilde{n},\tilde{\alpha}} \mp \frac{1}{\hat{t}_1^2} \partial_X \hat{t}_1 \hat{h}_{\tilde{n},\tilde{\alpha}} \pm \frac{2}{\hat{t}_1^3} \partial_X \partial_i \hat{t}_1 \hat{h}_{\tilde{n},\tilde{\alpha}} \mp \frac{1}{\hat{t}_1^2} \partial_i \hat{t}_1 \partial_X \hat{h}_{\tilde{n},\tilde{\alpha}}. \end{aligned} \quad (5.4.40)$$

Note the use of the identity $\hat{\nabla}_X \partial_i \hat{t}_1 = 0$, which may be interpreted as the fact the $\hat{\nabla}$ is a metric connection, or that the function \hat{t}_1 is a flat coordinate for the metric $\hat{\eta}$. We also have

$$\begin{aligned} \partial_X \log \hat{t}_1 \partial_i h_{\alpha,p} &= \partial_X \log \hat{t}_1 \left(\pm \frac{1}{\hat{t}_1} \partial_i \hat{h}_{\tilde{n},\tilde{\alpha}} \mp \frac{1}{\hat{t}_1^2} \partial_i \hat{t}_1 \hat{h}_{\tilde{n},\tilde{\alpha}} \right), \\ \partial_i \log \hat{t}_1 \partial_X h_{\alpha,p} &= \partial_i \log \hat{t}_1 \left(\pm \frac{1}{\hat{t}_1} \partial_X \hat{h}_{\tilde{n},\tilde{\alpha}} \mp \frac{1}{\hat{t}_1^2} \partial_X \hat{t}_1 \hat{h}_{\tilde{n},\tilde{\alpha}} \right). \end{aligned}$$

This means, perhaps quite remarkably, that

$$\hat{\nabla}_X (\partial_i h_{\alpha,p}) + \partial_X \log \hat{t}_1 \partial_i h_{\alpha,p} + \partial_i \log \hat{t}_1 \partial_X h_{\alpha,p} = \pm \frac{1}{\hat{t}_1} \hat{\nabla}_X \partial_i \hat{h}_{\tilde{n},\tilde{\alpha}}. \quad (5.4.41)$$

Finally we get

$$\nabla_X (\partial_i h_{\alpha,p}) = \pm \frac{1}{\hat{t}_1} \hat{\nabla}_X (\partial_i \hat{h}_{\tilde{n},\tilde{\alpha}}) \mp \frac{1}{\hat{t}_1^2} \frac{\partial u_i}{\partial X} \partial_i \hat{t}_1 \hat{h}_{\tilde{\alpha},\tilde{p}-1}, \quad (5.4.42)$$

which, upon use of (5.4.29), verifies (5.4.31). The point of this approach is that it can now, at least in theory, be extended to compute the transformation properties of higher covariant derivatives of the sections $\{dh_{\alpha,p}\}$.

5.5 The Inversion Symmetry and the Principal Hierarchy at Genus One

We have now built a library of transformation properties of all the components in the Hamiltonian representation of the principal hierarchy. Therefore let us now state the main Theorems of this chapter.

Theorem 11. *The perturbation to the flows*

$$\frac{\partial u_i}{\partial T_{[1]}'^{n,\alpha}} = \varepsilon^2 \eta^{ii} \nabla_X \left(\frac{\delta \mathcal{H}_{n,\alpha}'}{\delta u_i} \right) + \varepsilon^2 \nabla_X \left(\sum_{s=1}^N f^{is} \nabla_X^2 \left(\frac{\partial h_{n,\alpha}}{\partial u_s} \right) + \frac{1}{2} \sum_{s=1}^N \nabla_X f^{is} \nabla_X \left(\frac{\partial h_{n,\alpha}}{\partial u_s} \right) \right) + \mathcal{O}(\varepsilon^4).$$

arising from the transformation

$$t_\alpha(T) = v_\alpha(T) + \varepsilon^2 \frac{\partial^2}{\partial T^{\alpha,0} \partial T^{1,0}} \left[\frac{1}{24} \log \det c_{\mu\nu\sigma} t_X^\sigma \right]_{t=v(T)} + \mathcal{O}(\varepsilon^4),$$

transforms under the inversion symmetry as

$$\frac{\partial u_i}{\partial T_{[1]}'^{n,\alpha}} = \hat{\sigma}_{\tilde{n},\tilde{\alpha};i}^{(1)} + \hat{\nabla}_X \hat{\sigma}_{\tilde{n},\tilde{\alpha};i}^{(2)} - \partial_X \log \hat{t}_1 \hat{\sigma}_{\tilde{n},\tilde{\alpha};i}^{(2)} + \sum_{s \neq i} \partial_s \log \hat{t}_1 \hat{\sigma}_{\tilde{n},\tilde{\alpha};s}^{(2)} \partial_X (u_s - u_i) + \mathcal{O}(\varepsilon^4),$$

where

$$\begin{aligned} \hat{\sigma}_{\tilde{n},\tilde{\alpha};i}^{(1)} &= \hat{t}_1^2 \hat{\eta}^{ii} \hat{\nabla}_X \left(\frac{\delta \hat{\mathcal{H}}_{\tilde{n},\tilde{\alpha}}'}{\delta u_i} \right) + 2 \hat{t}_1 \partial_X \hat{t}_1 \widehat{\Delta \mathcal{H}}_{\tilde{n},\tilde{\alpha}}' \pm \hat{t}_1^2 \hat{\eta}^{ii} \partial_X \hat{t}_1 \frac{\delta \widehat{\Delta \mathcal{H}}_{\tilde{n},\tilde{\alpha}}'}{\delta u_i} + \hat{t}_1 \partial_X \hat{t}_1 \hat{\eta}^{ii} \frac{\delta \hat{\mathcal{H}}_{\tilde{n},\tilde{\alpha}}'}{\delta u_i} \\ &\quad + \hat{t}_1 \sum_{s \neq i} \left(\pm \hat{t}_1 \frac{\delta \widehat{\Delta \mathcal{H}}_{\tilde{n},\tilde{\alpha}}'}{\delta u_s} + \frac{\delta \hat{\mathcal{H}}_{\tilde{n},\tilde{\alpha}}'}{\delta u_s} \right) \partial_X (u_s - u_i) \\ \hat{\sigma}_{\tilde{n},\tilde{\alpha};i}^{(2)} &= \sum_{s=1}^N (\hat{t}_1^4 \hat{h}^{is} + \hat{t}_1^3 \hat{\phi}^{is}) \left\{ \pm \frac{1}{\hat{t}_1} (\hat{\nabla}_X - 2 \partial_X \log \hat{t}_1) \hat{\nabla}_X (\partial_s \hat{h}_{\tilde{n},\tilde{\alpha}}) \right. \\ &\quad \mp \frac{1}{\hat{t}_1^2} \hat{h}_{\tilde{n},\tilde{\alpha}} (\hat{\nabla}_X - 2 \partial_X \log \hat{t}_1 + \frac{1}{\hat{t}_1^2} \partial_X u_s) (\hat{\nabla}_X \partial_s \hat{h}_{1,1}) \pm \frac{1}{\hat{t}_1^2} \eta (\partial_X u, \partial_X u) (\partial_s \hat{h}_{\alpha,p-1}) \left. \right\} \\ &\quad \pm \frac{1}{2} \sum_{s=1}^N \left(\hat{t}_1^4 (\hat{\nabla}_X + 2 \partial_X \log \hat{t}_1) \hat{h}^{is} + 2 \hat{t}_1^3 (\hat{\nabla}_X + \partial_X \log \hat{t}_1) \hat{\phi}^{is} - \frac{N}{12} \hat{t}_1 (\partial_X (u_i + u_s) - \delta^{is} \eta^{ss} \partial_X \hat{t}_1) \right) \\ &\quad \times \left\{ \frac{1}{\hat{t}_1} \hat{\nabla}_X (\partial_s \hat{h}_{\tilde{n},\tilde{\alpha}}) - \hat{\nabla}_X (\partial_j \hat{h}_{1,1}) \hat{h}_{\tilde{n}-1,\tilde{\alpha}} \right\}, \end{aligned}$$

$\hat{f}^{ij}, \hat{\phi}^{ij}$ are as above, and

$$\tilde{n} = \begin{cases} n+1, & \text{if } \alpha = N, \\ n, & \text{if } \alpha \neq 1, N, \\ n-1, & \text{if } \alpha = 1, \end{cases} \quad \tilde{\alpha} = \begin{cases} 1, & \text{if } \alpha = N, \\ \alpha, & \text{if } \alpha \neq 1, N, \\ N, & \text{if } \alpha = 1, \end{cases} \quad \pm = \begin{cases} +, & \text{if } \alpha = N, \\ -, & \text{else.} \end{cases} \quad (5.5.1)$$

Theorem 12. *The perturbation to the flows*

$$\begin{aligned} \frac{\partial u_i}{\partial T_{[1]}'^{n,\alpha}} &= \varepsilon^2 \eta^{ii} \nabla_X \left(\frac{\delta \mathcal{H}_{n,\alpha}''}{\delta u_i} \right) \\ &+ \varepsilon^2 \nabla_X \left(\sum_{s=1}^N a^{is} \nabla_X^2 \left(\frac{\partial h_{n,\alpha}}{\partial u_s} \right) + \left(\frac{1}{2} \sum_{s=1}^N \nabla_X a^{is} + b^{is} \right) \nabla_X \left(\frac{\partial h_{n,\alpha}}{\partial u_s} \right) \right) + \mathcal{O}(\varepsilon^4). \end{aligned}$$

arising from the transformation

$$t_\alpha(T) = v_\alpha(T) + \varepsilon^2 \frac{\partial^2}{\partial T^{\alpha,0} \partial T^{1,0}} [G(t)]_{t=v(T)} + \mathcal{O}(\varepsilon^4),$$

transforms under the inversion symmetry as

$$\frac{\partial u_i}{\partial T_{[1]}'^{n,\alpha}} = \hat{\varepsilon}_{\tilde{n},\tilde{\alpha};i}^{(1)} + \hat{\nabla}_X \hat{\varepsilon}_{\tilde{n},\tilde{\alpha};i}^{(2)} - \partial_X \log \hat{t}_1 \hat{\varepsilon}_{\tilde{n},\tilde{\alpha};i}^{(2)} + \sum_{s \neq i} \partial_s \log \hat{t}_1 \hat{\varepsilon}_{\tilde{n},\tilde{\alpha};s}^{(2)} \partial_X (u_s - u_i) + \mathcal{O}(\varepsilon^4),$$

where

$$\begin{aligned} \hat{\varepsilon}_{\tilde{n},\tilde{\alpha};i}^{(1)} &= \hat{t}_1^2 \hat{\eta}^{ii} \hat{\nabla}_X \left(\frac{\delta \hat{\mathcal{H}}_{\tilde{n},\tilde{\alpha}}''}{\delta u_i} \right) + 2\hat{t}_1 \partial_X \hat{t}_1 \widehat{\Delta \mathcal{H}}_{\tilde{n},\tilde{\alpha}}'' \pm \hat{t}_1^2 \hat{\eta}^{ii} \partial_X \hat{t}_1 \frac{\delta \widehat{\Delta \mathcal{H}}_{\tilde{n},\tilde{\alpha}}''}{\delta u_i} + \hat{t}_1 \partial_X \hat{t}_1 \hat{\eta}^{ii} \frac{\delta \hat{\mathcal{H}}_{\tilde{n},\tilde{\alpha}}''}{\delta u_i} \\ &+ \hat{t}_1 \sum_{s \neq i} \left(\pm \hat{t}_1 \frac{\delta \widehat{\Delta \mathcal{H}}_{\tilde{n},\tilde{\alpha}}''}{\delta u_s} + \frac{\delta \hat{\mathcal{H}}_{\tilde{n},\tilde{\alpha}}''}{\delta u_s} \right) \partial_X (u_s - u_i) \\ \hat{\varepsilon}_{\tilde{n},\tilde{\alpha};i}^{(2)} &= \sum_{s=1}^N (\hat{t}_1^4 \hat{a}^{is} + \hat{t}_1^3 \hat{\psi}^{is}) \left\{ \pm \frac{1}{\hat{t}_1} (\hat{\nabla}_X - 2\partial_X \log \hat{t}_1) \hat{\nabla}_X (\partial_s \hat{h}_{\tilde{n},\tilde{\alpha}}) \right. \\ &\quad \mp \frac{1}{\hat{t}_1^2} \hat{h}_{\tilde{n},\tilde{\alpha}} (\hat{\nabla}_X - 2\partial_X \log \hat{t}_1 + \frac{1}{\hat{t}_1^2} \partial_X u_s) (\hat{\nabla}_X \partial_s \hat{h}_{1,1}) \pm \frac{1}{\hat{t}_1^2} \eta(\partial_X u, \partial_X u) (\partial_s \hat{h}_{\alpha,p-1}) \Big\} \\ &\quad \pm \frac{1}{2} \sum_{s=1}^N \left(\frac{1}{2} (\hat{t}_1^4 (\hat{\nabla}_X + 2\partial_X \log \hat{t}_1) \hat{a}^{is} + \hat{t}_1^2 \partial_X \hat{t}_1 \hat{\psi}^{is}) + \hat{t}_1^4 \hat{b}^{is} + \hat{\omega}^{is} \right) \\ &\quad \times \left\{ \frac{1}{\hat{t}_1} \hat{\nabla}_X (\partial_s \hat{h}_{\tilde{n},\tilde{\alpha}}) - \hat{\nabla}_X (\partial_j \hat{h}_{1,1}) \hat{h}_{\tilde{n}-1,\tilde{\alpha}} \right\}. \end{aligned}$$

and $\hat{a}^{ij}, \hat{b}^{ij}, \hat{\psi}^{ij}, \hat{\omega}^{ij}, \tilde{n}$ and $\tilde{\alpha}$ are as above.

Remarks. Note that schematically both theorems state that the flows are mapped to themselves, up to a conformal factor and the addition of an appropriately defined vector field:

$$\frac{\partial u_i}{\partial T_{[1]}'^{n,\alpha}} = \pm \hat{t}_1^3 \frac{\partial u_i}{\partial \hat{T}_{[1]}'^{\tilde{n},\tilde{\alpha}}} + \hat{v}_{\tilde{n},\tilde{\alpha};i}.$$

One drawback of the approach presented by Dubrovin & Zhang is that the complexity of the expressions grows rapidly; for the modular Frobenius manifold first presented in

Example 5 even the perturbations $\delta \mathcal{H}'_{1,\alpha}$, $\delta \mathcal{H}''_{1,\alpha}$ are becoming unmanageable. Secondly, as with their expression for the genus two free energy presented in [22], the scope of the Theorems 11, 12 remains at this stage abstract: we are restricted to computing examples where the canonical coordinates are known explicitly. Such examples also happen to be those for which the perturbations take on an extremely simple form, and a lot of the structure derived above is not present.

Let us consider a slightly different approach that could be used to independently verify the results of this chapter. Liu, Xu, & Zhang [41] computed the transformation properties of the genus one free energy,

$$\hat{\mathcal{F}}^{[1]}(\hat{\mathbf{t}}, \partial_X \hat{\mathbf{t}}) = \mathcal{F}^{[1]}(\mathbf{t}, \partial_X \mathbf{t}) - \frac{N}{24} \log t_1. \quad (5.5.2)$$

Dubrovin & Zhang's deformed Hamiltonians are given by [23]

$$\delta \mathcal{H}'_{n,\alpha} + \mathcal{H}''_{n,\alpha} = \frac{\partial^2 \mathcal{F}^{[1]}(\hat{\mathbf{t}}, \partial_X \hat{\mathbf{t}})}{\partial X \partial T^{n,\alpha}}.$$

Thus using the transformation properties of the vector fields given in Proposition (16), combined with the relation (5.5.2) one could independently verify the results of Propositions (19) and (20). The drawback of this approach is that tackling the transformation properties of the Poisson brackets in this framework appears to be more vexatious.

Outlook

The main theme of this thesis has been to study how the inversion symmetry singles out a special class of solution to the WDVV equations: modular Frobenius manifolds. In addition to the specific points raised earlier, there are two major avenues for further research that stem from this work:

1. Classification of modular Frobenius manifolds;
2. Construction of dispersive evolution equations with modular symmetry.

More specifically, in Chapter 2 we defined, and set out a program for classification of *modular Frobenius manifolds*. We used the homogeneity and modular properties of the prepotential to construct an ansatz for the WDVV equations. As the dimension of the Frobenius manifold increases the number of terms present in our ansatz grows rapidly (particularly for the homogeneous cases), and finding all the solutions of WDVV for a given ansatz becomes computationally unmanageable. An obvious avenue for further exploration would be to try to refine our ansatz by using not just the invariance under the inversion symmetry, but the full $SL(2, \mathbb{Z})$ action. It may also be of interest to place constraints on the behaviour of the pivot functions (which define the Rankin derivative in the corresponding modular dynamical system) at $\tau = i\infty$, and obtain classification results for modular Frobenius manifolds with specified analytic properties. It is also worth pointing out that all the modular Frobenius manifolds in the literature have an interesting level of symmetry between the polynomial variables. Perhaps this can be explained by more sophisticated representation theoretic techniques.

All the systems found in Chapter 2 are of rank 3, and are equivalent (sometimes via a non-linear change of variables) to a quadratic system. As already mentioned, if progress towards classification is to be made in higher dimensions is to be made, the reasons for this must be understood. Ohyama [49] has shown how a given quadratic system of rank 3 is naturally associated to a second order Fuchsian ODE (like the hypergeometric equation). It would be interesting to see if one can find explicit formulas for the structure constants of these quadratic systems in terms of the structure constants of the Frobenius algebras.

It would also be interesting to try and construct the almost dual prepotentials for the modular Frobenius manifolds of Chapter 2. The almost dual solutions should fall within the class found by Riley & Strachan [51]. Explicit verification of this statement for the modular Frobenius manifolds of Chapter 2 would help to provide understanding of the geometric origins of these Frobenius manifolds. An associated problem would be to construct the Landau-Ginzburg superpotentials for these solutions.

The action of the inversion symmetry on the dispersionless principal hierarchy is now completely understood: it acts as a reciprocal transformation. This reciprocal transformation should also be present at the level of the fully dispersive hierarchies, and it would be nice to obtain some explicit results. Chapter 5 shows how the flows behave at genus one, and in particular the results depend on the transformation properties of $\mathcal{F}^{[1]}$. In their work [41], Zhang et. al. presented a conjecture for how the genus- g generating functions $\mathcal{F}^{[g]}$ transform under the inversion symmetry for all $g \geq 0$. Understanding the action of the inversion symmetry on Dubrovin & Zhang's *universal loop equation* ([24], Theorem 3.10.31) would be key to proving this conjecture, and in turn perhaps allow one to obtain explicit results for how the fully dispersive hierarchies are related.

Bibliography

- [1] Atiyah, M., *Topological Quantum Field Theories*, Publ. Math. I.H.E.S. **68** (1988) 175.
- [2] Atiyah, M., and Hitchin, N., *The Geometry and Dynamics of Magnetic Monopoles*, Princeton University Press, Princeton, 1988.
- [3] Arnold, V.I. *Mathematical Methods of Classical Mechanics*, (Graduate Texts in Mathematics; 60) Springer-Verlag New York, Inc., 1978.
- [4] Arnold, V.I., Varchenko, A.N., and Gusein-Zade, S.M. *Singularities of Differentiable Maps, Volume 1*, Birkhäuser Boston, 1988, **ISBN** 0-8176-3185-2
- [5] Beilinson, A. and Levin, A., *The Elliptic Polylogarithm*, in *Motives* (ed. Jannsen, U., Kleiman, S., Serre, J.-P.), Proc. Symp. Pure Math. vol **55**, Amer. Math. Soc., (1994), Part 2, 123-190.
- [6] Bertola, M. *Frobenius manifold structure on orbit space of Jacobi groups; Parts I and II*, Diff. Geom. Appl. **13**, (2000), 19-41 and **13**, (2000), 213-23, and *Jacobi groups, Jacobi forms and their applications*, Ph.D. thesis, SISSA, 1999.
- [7] Bureau, F.J. *Sur des systèmes différentiels du troisième ordre et les équations différentielles associées*, Bulletin de la Classe des Sciences **73** (1987) 335–353.
- [8] Brini, A. *The Local Gromov-Witten Theory of \mathbb{CP}^1 and Integrable Hierarchies*, arXiv:1002.0582v3 [math-ph].
- [9] Chang, J.H. and Tu, M.H., *On the Benney Hierarchy: Free Energy, String Equation and Quantization*, J. Phys. A: Math. Gen. **34** 251 doi:10.1088/0305-4470/34/2/304.
- [10] Clarkson, P.A. and Olver, P.J., *Symmetry and the Chazy Equation*, J. Diff. Eqn. **124**, (1996), 225-246.
- [11] Coates, T., Corti, A., Lee, Y-P. and Tseng, H-H. *The Quantum Orbifold Cohomology of Weighted Projective Spaces*, Acta Mathematica, 202 (2009), 139-193.
- [12] Dijkgraaf, R., *A Geometrical Approach to Two-Dimensional Conformal Field Theory*, Ph.D. Thesis (Utrecht, 1989).

-
- [13] Dijkgraaf, R., Verlinde, H., and Verlinde, E., *Topological Strings in $d = 1$* , Nucl. Phys. B **352**59-86, 1991.
- [14] Dorfman, I., *Dirac Structures and Integrability of Nonlinear Evolution Equations*, John Wiley & Sons, 1993, **ISBN** 0471938939.
- [15] Drazin, P.G. and Johnson, R.S., *Solitons: An Introduction*, Cambridge University Press, Cambridge, 1989.
- [16] Dubrovin, B., *Geometry of 2D topological field theories in Integrable Systems and Quantum Groups. Montecatini, Terme 1993*, ed. Francaviglia, M. and Greco, S., Springer lecture notes in mathematics, **1620**, Springer-Verlag 1996, 120-348.
- [17] Dubrovin, B., *Integrable Systems in Topological Field Theory*, Nuclear Physics B 379 (1992) 627-689.
- [18] Dubrovin, B., *Flat pencils of metrics and Frobenius manifolds*, Proceedings of 1997 Taniguchi Symposium "Integrable Systems and Algebraic Geometry", Editors M.-H. Saito, Y. Shimizu and K. Ueno, 47-72. World Scientific, 1998.
- [19] Dubrovin, B., *Painlevé transcendents in two-dimensional topological field theory*, In: "The Painlevé property: one century later", Conte, R. (Ed.), Springer Verlag, 1999, p.287-412.
- [20] Dubrovin, B., *On almost duality for Frobenius manifolds in Geometry, topology, and mathematical physics*, 75-132, Amer. Math. Soc. Transl. Ser. 2, 212, Amer. Math. Soc., Providence, RI, 2004.
- [21] Dubrovin, B., and Novikov, S.P., *Hydrodynamics of Weakly Deformed Soliton Lattices, Differential Geometry and Hamiltonian Theory*, Uspekhi Mat. Nauk, 1989, v.44, N 6, 29-98.
- [22] Dubrovin, B. and Zhang, Y., *Extended affine Weyl groups and Frobenius manifolds*, Compositio Math. **111** (1998), 167-219.
- [23] Dubrovin, B. and Zhang, Y., *Bihamiltonian Hierarchies in 2D Topological Field Theory At One-Loop Approximation*, Commun. Math. Phys. **198** (1998) 311-361.
- [24] Dubrovin, B. and Zhang, Y., *Normal Forms of Integrable PDEs, Frobenius Manifolds and Gromov-Witten Invariants*, SISSA Preprint 65/2001/FM.
- [25] Dubrovin, B., Liu, S.-Q., and Zhang, Y., *On the Genus Two Free Energies for Semisimple Frobenius Manifolds*, arXiv: 1205.5990 [math-ph].

- [26] Dunin-Barkowski, P., Shadrin, S., and Spitz, P., *Givental Graphs and Inversion Symmetry*, arXiv:1201.4930v1 [math-ph].
- [27] Ferapontov, E.V., and Pavlov, M., *Reciprocal transformations of Hamiltonian operators of hydrodynamic type: nonlocal Hamiltonian formalism for linearly degenerate systems*, J. Math. Phys. 44 (2003), no. 3, 1150-1172.
- [28] Feigin, M., and Veselov, A., *Coxeter discriminants and logarithmic Frobenius structures*, Advances in Math., 2007, 212(1), pp.143-162.
- [29] Getzler, E., *Intersection Theory on $\bar{\mathcal{M}}_{1,4}$ and Elliptic Gromov-Witten Invariants*, J. Amer. Math. Soc. 10 (1997), no. 4, 973-998.
- [30] Goursat, E., *Sur L'Équation différentielle linéaire qui admet pour intégrale la série hypergéométrique*, Ann. Sci. de l'École Normale Supérieure **X** suppl., 1-142 (1881).
- [31] Geurts, M. L., Martini, R., Post, G.F., *Symmetries of the WDVV equations*, Acta Appl. Math. 72 (2002), no. 1-2, 67-75.
- [32] Guest, M., *From Quantum Cohomology to Integrable Systems*, Oxford Graduate Texts in Mathematics 15, Oxford University Press Inc., New York, 2008. **ISBN** 978-0-19-856599-4.
- [33] Halphen, G.-H., *Sur un Systèms D'équations Différentielles*, C.R. Acad. Sc. Paris **92** (1881) 1001-1003, 1004-1007.
- [34] Hertling, C., *Frobenius manifolds and moduli spaces for singularities*, Cambridge Tracts in Mathematics, Cambridge University Press, 2002.
- [35] Hitchin, N., *Frobenius Manifolds*, in "Gauge Theory and Symplectic Geometry", pages 69-112, Kluwer Acad. Publ., Dordrecht, 1997.
- [36] Humphreys, J.E., *Reflection Groups and Coxeter Groups*, Cambridge University Press, 1990. **ISBN** 0 521 43613 3.
- [37] Kock, J., *Frobenius Algebras and 2D Topological Quantum Field Theories*, Cambridge University Press, 2003.
- [38] Kontsevich, M., *Intersection Theory on the Moduli Space of Curves and the Matrix Airy Function*, Comm. Math. Phys. **147**, (1992), 1-23.
- [39] Lawrence, M., *Quadratic Differential Equations and Non-Associative Algebras*, Contributions to the theory of nonlinear oscillations, Vol. V pp. 185-213 Princeton Univ. Press (1960), Princeton, N.J..

-
- [40] Lechtenfeld, O. and Polovnikov, K., *A new class of solutions to the WDVV equation*, Phys. Lett. **A 374**, (2010), 504-506.
- [41] Liu, S.-Q., Xu, D. and Zhang, Y., *The inversion symmetry of the WDVV equations and tau functions*, Physica D, <http://dx.doi.org/10.1016/j.physd.2012.02.018>.
- [42] Maassarani, Z., *On the solutions of topological Landau-Ginzburg models with $c = 3$* , Phys. Lett. **B 273** (1991) 457-462.
- [43] Harnad, T. and McKay J., *Modular Solutions to Equations of Generalized Halphen Type*, Proc. R. Soc. Lond. A (2000) **456** 261-294.
- [44] Morrison, E.K. and Strachan, I.A.B., *Modular Frobenius Manifolds and Their Invariant Flows*, International Mathematics Research Notices (2010), article ID rnm236, 26 pages, doi:10.1093/imrn/rnq236
- [45] Morrison, E.K. and Strachan, I.A.B., *Polynomial Modular Frobenius Manifolds*, Physica D (2012), doi:10.1016/j.physd.2011.12.006
- [46] Milanov, T. and Ruan, Y., *Gromov-Witten theory of elliptic orbifold \mathbb{P}^1 and quasi-modular forms*, arXiv 1106:2321.
- [47] Noumi, M. and Yamada, Y., *Notes on the flat structures associated with simple and simply elliptic singularities*, in "Integrable Systems and Algebraic Geometry" (eds. M.-H. Saito, Y. Shimizu, K. Ueno), Proceedings of the Taniguchi Symposium 1997, 373-383, World-Scientific, 1998.
- [48] Novikov, S.P., *Solitons and Geometry*, Accademia Nazionale dei Lincei, 1994.
- [49] Ohyaama, Y., *Hypergeometric functions and non-associative algebras*, Proceedings on Moonshine and related topics (Montréal, QC, 1999), 173184, CRM Proc. Lecture Notes, 30, Amer. Math. Soc., Providence, RI, 2001.
- [50] Rankin, R., *Modular Forms and Functions*, Cambridge University Press, 1977, **ISBN 9780521212120**.
- [51] Riley, A. and Strachan, I.A.B., *Duality for Jacobi Group Orbit Spaces and Elliptic Solutions of the WDVV Equations*, Lett. Math. Phys. **77:3** (2006) 221-234.
- [52] Rogers, C., *Reciprocal Relations in Non-Steady One-Dimensional Gasdynamics*, Zeit angew. Math. Phys., 19, 58-63 (1968).
- [53] Rossi, P., *Gromov-Witten theory of orbicurves, the space of tri-polynomials and Symplectic Field Theory of Seifert fibrations*, Mathematische Annalen, **348(2)** (2010), 265-287.

- [54] Saito, K., *On a linear structure of a quotient variety by a finite reflection group*, Preprint RIMS-288 (1979).
- [55] Saito, K., *Extended Affine Root Systems I (Coxeter transformations), II (Flat invariants)*, Publ. RIMS, Kyoto Univ. **21** (1985), 75-179, **26** (1990), 15-78.
- [56] Satake, I., *Flat Structure for the Simple Elliptic Singularity of type \tilde{E}_6 and Jacobi Form*, Proc. Japan Acad. **69** Ser. A (1993) 247-251.
- [57] Satake, I., *Frobenius manifolds for elliptic root systems*, arXiv:math/0611553.
- [58] Satake, I., *Flat Structure and the Prepotential for the Elliptic Root System of Type $D_4^{(1,1)}$ in Topological Field Theory, Primitive Forms and Related Topics* (Progress in Mathematics vol. 160, Birkhäuser), ed. Kashiwara, M., Saito, K., Satake, I..
- [59] Satake, I. and Takahashi, A., *Gromov-Witten invariants for mirror orbifolds of simple elliptic singularities*, arXiv:1103.0951.
- [60] Satake, I. and Takahashi, A., *Private communication and forthcoming paper of Ebeling and Takahashi*.
- [61] Strachan, I.A.B., *Weyl groups and elliptic solutions of the WDVV equations*, Advances in Mathematics **224:5**, (2010), 1801-1838.
- [62] Strachan, I.A.B., *Simple Elliptic Singularities: a note on their G-function*, arXiv:1004.2140, to appear, *The Asian Journal of Mathematics*.
- [63] Strachan, I.A.B., *Symmetries and Solutions of Getzler's equation for Coxeter and Extended Affine Weyl Frobenius manifolds*, Int. Math. Res. Notices **19** (2003) 1035-1051.
- [64] Tsarev, S.P., *Poisson Brackets and Hamiltonian Systems of the Hydrodynamic Type*, Doklady Akademi Nauk SSSR, vol. 282, issue 3, (1985), 534-537.
- [65] Tu, M.H., *On dispersionless Lax representations for two-primary models*, J. Phys. A: Math. Theor. **40** 9427.
- [66] Verlinde, E. and Warner, N.P., *Topological Landau-Ginzburg matter at $c = 3$* , Phys. Lett. **B 269** (1991) 96-102.
- [67] Witten, E., *Two Dimensional Gravity and Intersection Theory on the Moduli Space*, Surveys in Differential Geometry vol. 1, 1991.
- [68] Whittaker, E.T. and Watson, G.N., *A Course of Modern Analysis*, Cambridge University Press, 1962.

-
- [69] Yoshida, M., *Fuchsian Differential Equations With Special Emphasis on Gauss-Schwarz Theory*, Friedr, Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig (1987).
- [70] Zuber, J.-B., *On Dubrovin Topological Field Thoeries*, Mod. Phys, Lett, **A9** (1994) 749-760.